Introduction to Combinatorics

Homework 8, due date: 2020-05-21

1. Let $c > 0$ be a constant and let $p_n = c\frac{\ln n}{n}$. Consider an event $\mathcal{C}_n = \{G(n, p_n) \text{ has no isolated vertices}\}$.

   (a) Show that if $c < 1$ then $\mathbb{P}(\mathcal{C}_n) \to 0$ as $n \to \infty$.

   (b) Show that if $c > 1$ then $\mathbb{P}(\mathcal{C}_n) \to 1$ as $n \to \infty$.

2. Let $k, d \geq 1$ be integers. Let $\mathcal{F}$ be a family of k-subsets of a give set X. Suppose that every point in X belongs to at most d members of $\mathcal{F}$. Show that there exists a function $f : X \to \{-1, 1\}$ such that for any set $A \in \mathcal{F}$ we have

$$\left| \sum_{a \in A} f(a) \right| \leq \sqrt{2k \log(2ekd)}.$$

3. Prove that $p_n := \frac{\ln n}{n}$ is a threshold function for connectivity of $G(n, p_n)$.

4. Suppose $f : \{-1, 1\}^n \to \mathbb{R}$ is not identically zero. Let $f = \sum_S a_S w_S$ be its Fourier expansion and let $\deg(f) = \max\{|S| : \ a_S \neq 0\}$. Show that $|\{x : \ f(x) \neq 0\}| \geq 2^{n-\deg(f)}$.

5. Suppose $f : \{-1, 1\}^n \to \{-1, 1\}$ depends on all its variables (for any i there exists at least one point x such that flipping the ith bit of x changes the value $f(x)$). Let $f = \sum_S a_S w_S$ be its Fourier expansion and let $d = \max\{|S| : \ a_S \neq 0\}$. Show that $n \leq d2^d$. In particular $d \geq \frac{1}{2} \ln n$.

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6. Let G_n be the hypercube graph, namely $G_n = (\{0, 1\}^n, E)$, where $x \sim y$ if $|x - y| = 1$. Show that for any $A \subseteq \{-1, 1\}^n$ we have $|\partial A| \geq |A|(n - \log_2 |A|)$.

7. Show that there exists $f : \{-1, 1\}^n \to \{-1, 1\}$ depending on all its variables (for any i there exists at least one point x such that flipping the ith bit of x changes the value $f(x)$) and such that $\deg(f) = O(\ln n)$.