Introduction to Combinatorics

Homework 7, due date: 2020-04-30

1. Let G be a graph with n vertices and m edges. Prove that we can partition $V(G)$ into A and B such that there are at least $(1+\frac{1}{n})\cdot\frac{m}{2}$ edges with one endpoint in A and one in B.

2. Let G be a graph so that all its vertices have degree at least d. Prove that there exists a dominating set of G of size at most $n\frac{1+\ln(1+d)}{1+d}$.

   Definition: D is a dominating set of G if and only if every vertex $v \in V(G)$ either is in D or has a neighbour in D.

3. There are $3n$ people that speak in n languages. Each of them speaks exactly 3 of these languages. Prove that there exists a set A of at least $\frac{2}{9}n$ languages so that for every guy there exists a language he speaks that is not in A.

4. Let S be a finite set of $n \geq 3$ points on a plane such that no three of them are collinear. For every convex polygon P with vertices in S, let $a(P)$ denote number of its vertices and $b(P)$ denote number of points in S that lie strictly outside of P. Prove that:

$$\sum_P \pi^{a(P)}(1-\pi)^{b(P)} = 1 - \sum_{k=0}^{2}\binom{n}{k}\pi^k(1-\pi)^{n-k},$$

   where this is sum over all convex polygons with vertices in S with at least three vertices.

5. Suppose that an $n \times n$ array of lights is given. Each light is on or off. Suppose for each row and each column there is a switch so that if the switch is pulled all of the lights in that line are "switched": on to off or off to on. Show that for any initial configuration it is possible to perform switches so that the number of lights on minus the number of lights off is at least $\left(\sqrt{\frac{2}{\pi}} - o(1)\right)n^{3/2}$.

   Note: $\left(\sqrt{\frac{2}{\pi}} - o(1)\right)n^{3/2}$ in a more elaborate way means that for every $\varepsilon > 0$ for sufficiently large n this value is at least $\left(\sqrt{\frac{2}{\pi}} - \varepsilon\right)n^{3/2}$

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6. Show that for every bipartite $n \times n$ graph with $n \geq 3$ we have $\mathrm{ch}(G) \leq 2\log_2 n$. Show that this bound is optimal, up to the multiplicative constant.

   Definition: Let $G = (V, E)$ be a graph. The choice number $\mathrm{ch}(G)$ is the minimal number k such that for every assignment of a set $S(v)$ of k colors to every vertex v of G there is a choice $k_v \in S(v)$ of colors such that $\{u, v\} \in E$ implies $k_v \neq k_u$.

7. Let $z_1, z_2, \ldots, z_n$ be complex numbers. Prove that there exists a subset I of $\{1, 2, \ldots, n\}$ such that

$$\left|\sum_{k\in I} z_k\right| \geq \frac{1}{\pi}\sum_{k=1}^{n}|z_k|.$$

   Is the constant $1/\pi$ optimal?