Introduction to Combinatorics

Homework 6, due date: 2020-04-16

1. We are given an $n \times n$ board. Each of its cells should contain with an integer from 1 to n such that there are no equal numbers within any row or within any column. Somebody filled k first rows so that this condition holds and left remaining $n-k$ rows empty. Prove that we can fill the rest of this board such that mentioned condition is still true.

2. Let $G = (V, E)$ be a bipartite graph. Prove that it is possible to color edges of G in $\Delta(G)$ colors such that whenever two edges share a vertex, they have different colors and such that no color is used more than $\lceil \frac{|E|}{\Delta(G)} \rceil$ times (where $\Delta(G)$ denotes the biggest degree of a vertex in G).

3. Let $G = (V, E)$ be a directed graph. Call a set $S \subseteq V$ *closed* if and only if there is no edge (u, v) such that $u \in S$ and $v \notin S$. There are some tokens placed in vertices (there can be more than one token in each vertex). In one move we can take one token from some vertex v and move it to vertex u if $(v, u) \in E$. For every vertex v there are two nonnegative integer numbers a_v, b_v, where a_v is an initial number of tokens in vertex v and $\sum_{v \in V(G)} a_v = \sum_{v \in V(G)} b_v$. Prove that we can perform finite number of steps so that there are b_v tokens in vertex v for every vertex v if and only if there is no closed set S such that $\sum_{v \in S} a_v > \sum_{v \in S} b_v$.

4. Let G be an undirected graph (not necessarily bipartite). Consider a following game. There is a token put in a vertex v. There are two players and they take turns alternately. In each turn player must move token from a current vertex to an adjacent vertex u so that token was never in u before. If player can't make a move, he loses. Prove that first player has a winning strategy if and only if v belongs to every maximum matching in G.

5. There are n boys and n girls on a party. Every boy knows at least k girls and is willing to dance with them (willingness to dance is mutual). Let us assume that it is possible to form n dancing pairs such that in every pair boy and girl know each other. Prove that there are at least $k!$ different ways to form n dancing pairs.