Introduction to Combinatorics

Homework 5, due date: 2020-04-02

1. For an undirected graph G, let $mad(G)$ denote the maximum average degree of a subgraph of G, i.e.

$$mad(G) = \max_{H \subseteq G, H \neq \emptyset} \frac{2E(H)}{G(H)}.$$

   Let $g(G)$ denote length of the shortest cycle in G (called *girth*). Prove that if G is a planar graph with at least one cycle then $(mad(G) - 2)(g(G) - 2) < 4$.

2. Let d be an integer and G be an undirected connected graph on n vertices so that every vertex has degree at least d and $2d < n$. Prove that there exists a simple path with $2d + 1$ vertices in G.

3. Prove that there exists an absolute constant C such that any planar graph on n vertices has at most Cn cliques.

4. There are n points marked on the plane. Prove that we can color each point in either red or blue so that for every vertical or horizontal line, absolute value of difference between number of red points on it and blue points on it, is at most 1.

   Hint: Try similar approach as to the problem about pawns on chessboard from the exercises.

5. Let $G = (V, E)$ be a connected graph. Show that the following conditions are equivalent:

   (i) G is Eulerian,

   (ii) E can be decomposed into cycles[1]

   (iii) G has an odd number of cycle decompositions,

   (iv) every edge of G lies on an odd number of cycles.

6. Let G be a connected planar graph with only pentagonal and hexagonal faces (including the infinite face). Find all possible values of the number of pentagonal faces if:

   (a) G is assumed to be 3-regular,

   (b) G is not assumed to be 3-regular

[1] A cycle is a closed walk of the form $v_1 \to v_2 \to \ldots v_n \to v_1$, where v_i are different, with no repeated edges.