Introduction to Combinatorics

Homework 4, due date: 2020-03-26

1. Let p be a prime number, A be a subset of $\mathbb{Z}_p$. Prove that $|A \oplus A| \geq \min(p, 2|A| - 3)$, where $B \oplus C = \{b + c : b \in B, c \in C, b \neq c\}$.

2. Let $s(i,j) = \begin{cases} 1 & \text{if } i \leq j \\ -1 & \text{if } i > j \end{cases}$

   Determine the coefficient next to $x_1 x_2 \ldots x_{2020}$ in $\prod_{i=1}^{2020} (\sum_{j=1}^{2020} s(i,j) x_j)$

3. Erdős-Ginzburg-Ziv theorem says that for every positive integer n and every set of integers of size $2n - 1$ there exists a subset of this set, having cardinality n, so that the sum of its elements is divisible by n. In the exercises we proved it when n is a prime number. Conclude its general version.

4. Let n be a positive integer and $S = \{(x, y, z) : x, y, z \in \{0, \ldots, n\}, x + y + z > 0\}$. Find smallest number k so that there exist k planes whose sum contains S but doesn't contain $(0, 0, 0)$.

5. Let A be a finite nonempty subset of $\mathbb{Z}$. Show that $|A + A| \geq 2|A| - 1$. Moreover, show that if $|A + A| = 2|A| - 1$ then A is an arithmetic progression.

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6. A set $C \subseteq \mathbb{R}$ is called *nice* if C is a finite sum of closed **disjoint** intervals $I_i = [a_i, b_i]$, where $a_i < b_i$ for $i = 1, \ldots, n$ and $n \geq 1$. Then the *length* of C is defined as $|C| = \sum_{i=1}^{n} |b_i - a_i|$.

   Show that if A and B are nice sets, then the set $A + B = \{a + b : \ a \in A, b \in B\}$ is also nice and $|A + B| \geq |A| + |B|$.