Introduction to Combinatorics

Piotr Nayar, homework III, due date: 19/03/2020

1. Suppose $v_1, \ldots, v_n \in \mathbb{R}^d$ are such that $|v_i - v_j| = 1$ for $i \neq j$. Show that $n \leq d+1$. Is this bound sharp?

2. For $u = (u_1, \ldots, u_d) \in \mathbb{R}^d$ we define $\|u\|_\infty = \max_{i=1,\ldots,d} |u_i|$. Suppose $v_1, \ldots, v_m$ be vectors in $\mathbb{R}^d$ such that $\|v_i - v_j\|_\infty = 1$ for all $i \neq j$. Show that $m \leq 2^d$. Is this bound sharp?

3. Let $N = \frac{1}{2}n(n+1)$. Show that there exist distinct vectors $v_1, \ldots, v_N$ in $\mathbb{R}^n$ and real numbers $a, b > 0$ such that $|v_i - v_j| \in \{a, b\}$ for all $i \neq j$.

4. Let $B_2^d = \{x \in \mathbb{R}^d : |x| \leq 1\}$.
   (a) Show that B_2^d can be partitioned into $d+1$ sets of diameter smaller than 2.
   (b) Is there a partition of B_2^d into fewer than $d+1$ sets of diameter smaller than 2?

5. Let A be a finite subset of $[0, 1)$ and let l be the cardinality of $A \cup (-A)$. Suppose $v_1, \ldots, v_m$ are unit vectors in $\mathbb{R}^d$ such that $|\langle v_i, v_j \rangle| \in A$ for every $i \neq j$. Show that $m \leq \binom{d+l-1}{l}$.

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6. Let $u_1, u_2, \ldots, u_m$ be non-zero vectors in $\mathbb{R}^d$ satisfying the condition $\langle u_i, u_j \rangle \leq 0$ for all $i \neq j$. Show that $m \leq 2d$. Is this bound tight?

7. Let $\Delta(G)$ be the maximum degree of a vertex in G and let $\chi(G)$ be the chromatic number[1] of G. Prove that $\chi(G) \leq \Delta(G) + 1$.

8. Let G be a graph on n vertices, and $\alpha(G)$ be its independence number, i.e., the maximal number of vertices, no two of which are joined by an edge. Show that $\frac{n}{\alpha(G)} \leq \chi(G) \leq n - \alpha(G) + 1$.

9. Let $\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3$ be three basis of $\mathbb{R}^n$. Must there exist $v_i \in \mathcal{B}_i$ for $i = 1, 2, 3$ such that v_1, v_2, v_3 are pairwise non-orthogonal?

[1] Chromatic number of a graph $G = (V, E)$ is the smallest number of colors needed to color the vertices of G in such a way that no two adjacent vertices receive the same color.