Introduction to Combinatorics

Piotr Nayar, homework II, due date: 12/03/2020

1. Suppose $n_1 < n_2 < \ldots < n_{1000}$ be positive integers. Prove that one can either select 28 of them, no one of which divides any other, or 38 of them, each dividing the following one.

2. Let $r \geq 1$ be an integer. Let $\mathcal{F}$ be a family of distinct subsets of a given finite set, such that for every distinct $A, B \in \mathcal{F}$ we have
$$|A \cap B| < \frac{1}{r} \min\{|A|, |B|\}.$$
Show that for all distinct $A_0, A_1, \ldots, A_r \in \mathcal{F}$ we have $A_0 \nsubseteq A_1 \cup A_2 \cup \ldots \cup A_r$.

3. Let $G = (V, E)$ be a graph on n vertices and let $t(G)$ be the number of triangles in it. Show that
$$t(G) \geq \frac{|E|}{3n}(4|E| - n^2).$$

4. A chain of subsets $A_1 \subset \ldots \subset A_k$ of an n element set is called *symmetric* if $|A_1| + |A_k| = n$ and $|A_{i+1}| = |A_i| + 1$ for $i = 1, \ldots, k-1$. Prove that the family of all subsets of a given n element set can be partitioned into $\binom{n}{[n/2]}$ disjoint symmetric chains.

5. Let $r \geq 1$ be an integer. Show that there exists an integer $n_0 = n_0(r)$ with the following property: whenever $n \geq n_0$ and we color all non-empty subsets of $\{1, \ldots, n\}$ with r colors, there exist disjoint non-empty subsets A and B, such that A, B and $A \cup B$ have the same color.