Introduction to Combinatorics

Homework – special I, due date: 2020-06-11

1. Let $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ be two sequences. Show that

$$\forall_{n\geq 0} \quad a_n = \sum_i \binom{n}{i}(-1)^i b_i \qquad \iff \qquad \forall_{n\geq 0} \quad b_n = \sum_i \binom{n}{i}(-1)^i a_i.$$

2. Let D_n be the number of permutations with no fixed points (*derangements*).
   (a) Show that $n! = \sum_i \binom{n}{i} D_i$.
   (b) Show that $D_n = n! \sum_{i=0}^{n} \frac{(-1)^i}{i!}$.

3. Let $\left[{n \atop k}\right]$ be the *Stirling number of the first kind*, that is, the number of permutations of $[n]$ with k cycles. Let $\left\{{n \atop k}\right\}$ be the *Stirling number of the second kind*, that is, the number of ways to partition the set $[n]$ into k unlabelled subsets.
   (a) Show that $\left[{n \atop k}\right] = (n-1)\left[{n-1 \atop k}\right] + \left[{n-1 \atop k-1}\right]$.
   (b) Show that $\left\{{n \atop k}\right\} = k\left\{{n-1 \atop k}\right\} + \left\{{n-1 \atop k-1}\right\}$.
   (c) Let $x^{\overline{n}} = x(x+1)\dots(x+n-1)$ and $x^{\underline{n}} = x(x-1)\dots(x-n+1)$. Show that we have $x^n = \sum_k \left\{{n \atop k}\right\} x^{\underline{k}}$ and $x^{\overline{n}} = \sum_k \left[{n \atop k}\right] x^k$.
   (d) Prove that $\sum_i \left\{{n \atop i}\right\}\left[{i \atop k}\right](-1)^{n-i} = \delta_{n=k}$ and $\sum_i \left[{n \atop i}\right]\left\{{i \atop k}\right\}(-1)^{n-i} = \delta_{n=k}$. Here δ stands for the Kronecker delta.

4. Let B_n be the number of partitions of $[n]$ into blocks, namely $B_n = \sum_k \left\{{n \atop k}\right\}$ (*Bell numbers*).
   (a) Show that $B_{n+1} = \sum_k \binom{n}{k} B_k$.
   (b) Define $B(x) = \sum_n \frac{B_n}{n!} x^n$. Show that $B'(x) = e^x B(x)$ and find $B(x)$.
   (c) Show that $B_n = \frac{1}{e}\sum_{k=0}^{\infty} \frac{k^n}{k!}$.

5. Let R_n be the number of ways to write n as a sum of pairwise different non-negative integers. Let S_n be the number of ways to write n as a sum of odd integers (permutation of numbers gives the same division). Using generating functions show that $R_n = S_n$.

6. Let P_n be the number of ways to write n as a sum of positive summands, written in a non-increasing order (*partitions*). Show that $nP_n = \sum_{k=0}^{n-1} \sigma(n-k)P_k$, where $\sigma(n) = \sum_{k|n} k$.

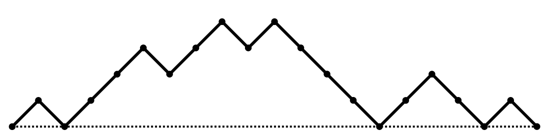
7. (Inclusion-exclusion principle) Let $A_1, \dots, A_n$ be subsets of a finite set X. Let $S_j = \sum_{1\leq i_1 < i_2 < \dots < i_j \leq n} |A_{i_1} \cap A_{i_2} \cap \dots \cap A_{i_j}|$ and $S_0 = |X|$. Let D_k be the number of elements of X belonging to precisely k subset A_i. Show that $D(k) = \sum_{j\geq k} \binom{j}{k}(-1)^{j-k} S_j$. In particular $D(0) = \sum_{j\geq 0}(-1)^j S_j$.

   Show that the number of surjective functions $f : \{1, \dots, r\} \to \{1, \dots, n\}$ equals $\sum_{j=0}^{n}(-1)^j\binom{n}{j}(n-j)^r$.

8. Let $a_0 = -1, a_1 = -5, a_2 = -5, a_3 = 2$ and $a_{n+4} = 3a_{n+3} - a_{n+2} - 4a_n$ for $n \geq 0$. Find a_n.

   Remark. Give the final result using only real numbers. Answer: $a_n = 2\cos(\frac{2}{3}\pi n) + 2^n(n-3)$.

9. Let C_n be the number of lattice paths from $(0,0)$ to $(2n,0)$ consisting of moves $(1,1)$ and $(1,-1)$ and staying above the x-axis (see the example below). Show that $C_n = \frac{1}{n+1}\binom{2n}{n}$.



10. Prove the following identities (for most of them it is preferable to give combinatorial argument):

(a) $\sum_{j=0}^{k}(-1)^j\binom{n}{j} = (-1)^k\binom{n-1}{k}$ (b) $\sum_{k=0}^{n} k^{\underline{l}}\binom{n}{k} = n^{\underline{l}}2^{n-l}$. (c) $\sum_k \binom{n}{k}\binom{m}{l-k} = \binom{n+m}{l}$
(d) $\sum_{k=m}^{n}\binom{k}{m} = \binom{n+1}{m+1}$ (e) $\sum_{k=0}^{m}\binom{n+k}{k} = \binom{n+m+1}{m}$ (f) $\sum_{k=0}^{[n/2]}\binom{n-k}{k} = F_{n+1}$

Here F_n is the n-th Fibonacci number.