Introduction to Combinatorics

Homework 10, due date: 2020-06-10

1. A partition of $\{1, 2, \ldots, n\}$ into disjoint sets $A_1, \ldots, A_k$ is called *non-crossing* if for any i, j if $a < b < c < d$ and $a, c \in A_i$ and $b, d \in A_j$, then $i = j$. Show that the number of non-crossing partitions of $\{1, 2, \ldots, n\}$ is the nth Catalan number.

2. A Ferrers diagram is *self-conjugate* if the number of cells in the ith column is equal to the number of cells in the ith row. Show that the number of self-conjugate Ferrers diagrams with n cells is equal to the number of ways of writing n as a sum of distinct positive odd integers (order doesn't matter).

3. Let $\mathcal{H}$ be a finite family of closed half-spaces in $\mathbb{R}^d$ and let C be a convex set in $\mathbb{R}^d$ such that $C \subseteq \bigcup_{H \in \mathcal{H}} H$. Show that there exists a subfamily $\mathcal{H}' \subseteq \mathcal{H}$ with $|\mathcal{H}'| \leq d + 1$ such that $C \subseteq \bigcup_{H \in \mathcal{H}'} H$.

4. (a) Let X be a subset of $\mathbb{R}^d$. Show that if every set $X_0 \subseteq X$ with $|X_0| = d + 1$ can be covered by a closed ball of radius r then X can be covered by a closed ball of radius r.
   (b) Let X be a subset of $\mathbb{R}^d$ of diameter at most 1. Show that X can be covered by a closed ball of radius at most $\sqrt{\frac{d}{2(d+1)}}$.

5. Let K, L be nonempty open convex sets in $\mathbb{R}^d$. Define $f(x) = \mathrm{vol}((K + x) \cap L)$.
   (a) Show that $f(x) > 0$ for $x \in L - K$.
   (b) Show that on the set $L - K$ the function $f^{1/d}$ is concave.

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6. For $a \in \mathbb{R}^l$, $r \geq 0$ let $B(a, r)$ be a closed ball centered at a with radius r.
   (a) Let $x_1, \ldots, x_k \in \mathbb{R}^n$ and $y_1, \ldots, y_k \in \mathbb{R}^m$ be such that $|y_i - y_j| \leq |x_i - x_j|$ for all i, j. Show that for any $r_1, \ldots r_k \geq 0$
   $$\bigcap_{i=1}^{k} B(x_i, r_i) \neq \emptyset \qquad \Longrightarrow \qquad \bigcap_{i=1}^{k} B(y_i, r_i) \neq \emptyset.$$
   (b) Suppose A is a finite set $\mathbb{R}^n$ and let $f : A \to \mathbb{R}^m$ has Lipschitz constant 1. Show that for any $x \in \mathbb{R}^n$ there exists a function $g : A \cup \{x\} \to \mathbb{R}^m$ such that $g\,|_A = f$ and g has Lipschitz constant 1.
   (c) Show that for any $A \subseteq \mathbb{R}^n$ and any 1-Lipschitz function $f : A \to \mathbb{R}^m$ there exists a 1-Lipschitz function $g : \mathbb{R}^n \to \mathbb{R}^m$ such that $g\,|_A = f$.

7. Let $A_1, \ldots, A_{n+1}$ be nonempty subsets of $\{1, \ldots, n\}$. Show that there exist disjoint subsets I, J of $\{1, \ldots, n, n + 1\}$ such that $\bigcup_{i \in I} A_i = \bigcup_{j \in J} A_j$.

8. Let us consider a symmetric random walk in $\mathbb{Z}^d$ starting from the origin, that is, in every step with probability $\frac{1}{2d}$ we go from the current point x to one of the $2d$ points $y \in \mathbb{Z}^d$ satisfying $|x - y| = 1$. Let $p_n(x)$ be the probability that our walk is in point x after n steps. Show that for any $x \in \mathbb{Z}^d$ and for any even integer n we have $p_n(0) \geq p_n(x)$.

9. Find all functions $f : \mathbb{Z}^2 \to [0, 1]$ satisfying the condition

$$f(x, y) = \frac{1}{4} \left(f(x-1, y) + f(x+1, y) + f(x, y-1) + f(x, y+1) \right), \qquad x, y \in \mathbb{Z}.$$

10. The goal of this exercise is to convince you that the procedure of deriving the formula for Catalan numbers presented during the lecture was correct even though we did not know that the generating function of Catalan numbers was analytic.

    Suppose that f is analytic in some neighborhood of zero $I = (-a, a)$ and that in I we have $f(x) = \sum_{i=0}^{\infty} a_i x^i$. Suppose that f satisfies $xf(x)^2 - f(x) + 1 = 0$ for $x \in I$. Prove that a_n satisfies $a_0 = 1$ and $a_{n+1} = \sum_{i=0}^{n} a_i a_{n-i}$ for all $n \geq 0$ and thus f is a generating function of Catalan numbers.

11. Suppose $\lambda_1, \ldots, \lambda_p$ are pairwise different complex numbers and let $l_1, \ldots, l_p$ be positive integers. Show that the following $l_1 + \ldots + l_p$ sequences

$$(\lambda_1^n)_{n\geq 0},\ (n\lambda_1^n)_{n\geq 0},\ \ldots,\ (n^{l_1-1}\lambda_1^n)_{n\geq 0}, \qquad \ldots \qquad (\lambda_p^n)_{n\geq 0},\ (n\lambda_p^n)_{n\geq 0},\ \ldots,\ (n^{l_p-1}\lambda_p^n)_{n\geq 0}$$

    are linearly independent. This exercise shows that the method of characteristic polynomials will always give a solution to the linear equation.