Introduction to Combinatorics

Piotr Nayar, homework I, due date: 05/03/2020

1. Let $A_1, \ldots, A_m$ and $B_1, \ldots, B_m$ be subsets of a given n element set, such that $|A_i \cap B_i|$ is odd for all i and $|A_i \cap B_j|$ is even for all $i \neq j$. Prove that $m \leq n$.

2. Let $n \leq 2k$ and let $A_1, \ldots, A_m$ be distinct k element subsets of a given set X with n elements. Suppose $A_i \cup A_j \neq X$ for all i, j. Show that $m \leq (1 - \frac{k}{n})\binom{n}{k}$.

3. Let $A_1, \ldots, A_m$ and $B_1, \ldots, B_m$ be subsets of a given finite set X such that $A_i \cap B_j = \emptyset$ if and only if $i = j$. Let $a_i = |A_i|$ and $b_i = |B_i|$. Prove the inequality
$$\sum_{i=1}^{m} \binom{a_i + b_i}{a_i}^{-1} \leq 1.$$
Deduce that if $A_1, \ldots, A_m$ are a element subsets and $B_1, \ldots, B_m$ are b element subsets of a given finite set X, such that $A_i \cap B_j = \emptyset$ if and only if $i = j$, then $m \leq \binom{a+b}{a}$. Is this bound tight?

4. Let $A_1, \ldots, A_m$ be distinct subsets of an n element set, such that $|A_i \cap A_j|$ is even for every $i \neq j$. Show that if $n > 5$ then $m \leq 2^{[n/2]}$ if n is even and $m \leq 2^{[n/2]} + 1$ if n is odd. Moreover, $m \leq n + 1$ if $n \leq 5$. Are these bounds tight?

5. Suppose that $\mathcal{F}$ is a family of distinct subsets of an n element set X, such that $|A|$ is even and $|A \cap B|$ is even for every $A, B \in \mathcal{F}$. Suppose also that no subset of X (not belonging to $\mathcal{F}$) can be added to $\mathcal{F}$ without violating the above parity rule. Show that $|\mathcal{F}| = 2^{[n/2]}$.

---

6. Let V be an N-dimensional vector space over arbitrary field and let U be an n-dimensional subspace of V. Let $v_1, \ldots, v_N$ be a basis for V. Show that U contains at most 2^n vectors of the form $\sum_{i=1}^{N} \lambda_i v_i$, where $\lambda_i \in \{0, 1\}$ for all i.