Introduction to Combinatorics

Charging and discharging

1 Examples

Example 1. A table has m rows and n columns, where $m < n$. Some cells of this table are marked in such a way that every column contains at least one marked cell. Prove that there is a marked cell such that the number of marked cells in its row is larger than the number of marked cells in its column.

Solution. Without loss of generality we can delete the empty rows, so we shall assume that in every row at least one cell is marked. To every column we give a charge -1 and to every row a charge 1. The total charge is $m - n < 0$. Now we discharge the row and columns in such a way that every column and every row distributes its charge evenly to each of its marked cells. After this procedure there is a marked cell with negative charge. This means that the number of marked cells in its column was strictly smaller than the number of marked cells in its row. □

Example 2. Suppose G is a simple planar graph with minimum degree at least 5. Then G contains an edge $\{x, y\}$ such that $\deg(x) + \deg(y) \leq 11$.

Solution. By adding edges we can assume that G is a triangulation (note that if we add edges the minimum degree only increases and if we find an edge $\{x, y\}$ such that $\deg(x)+\deg(y) \leq 11$ in the new graph, this edges will also be present in the original graph, since otherwise one of its endpoint would have degree smaller than 5 in the original graph).

Let V be the set of vertices, E the set of edges, and F the set of faces of our triangulation. Let $d(v)$ be the degree of $v \in V$ and let $d(f)$ be the number of edges on the boundary of f. Here, if $e \in E$ is on the boundary of only one face, then we count it two times. Hence, $\sum_f d(f) = 2|E| = \sum_{v \in V} d(v)$. Thus, by Euler's formula, we have

$$\sum_{v \in V}(6 - d(v)) = \sum_{f \in F}(6 - 2d(f)) + \sum_{v \in V}(6 - d(v)) = 6|F| - 4|E| + 6|V| - 2|E| = 12,$$

since for every face $d(f) = 3$. Let us give a charges $6 - d(v)$ to every vertex $v \in V$. The total charge is 12. The only vertices with positive initial charge are vertices satisfying $d(v) = 5$.

Now we discharge the system using a single rule: every vertex of degree 5 gives charge $\frac{1}{5}$ to each of its neighbors. Clearly the total final charge is still 12. Thus there are vertices with positive final charge. Suppose v is such a vertex. Then its final charge $c(v)$ satisfies $0 < c(v) \leq 6 - d(v) + \frac{1}{5}d(v) = 6 - \frac{4}{5}d(v)$. Thus $d(v) \leq 7$.

Now we consider three case.

Case 1. Suppose v with positive final charge satisfies $d(v) = 6$. Then its initial charge was 0 and thus this vertex must have gained some charge. So, one of its neighbors u has degree 5 and $\{u, v\}$ is the desired edge.

Case 2. Suppose v with positive final charge satisfies $d(v) = 5$. Then in the discharging process this vertex gave all its charge to its five neighbors. But since the final charge is positive, it must have gained some charge from a vertex of degree 5. Thus, one of its neighbors u has degree 5 and so $\{v, u\}$ is the desired edge.

Case 3. Suppose v we positive final charge satisfies $d(v) = 7$. Since the initial charge of v was -1, v gained charge from at least 6 of its neighbors. The neighbors of v (these are vertices of degree 5). There is one more neighbor of v whose degree we do not control. Since G is a triangulation, the neighbors of v form a cycle and clearly on this cycle there are two adjacent vertices u_1 and u_2 of degree 5. Thus $\{u_1, u_2\}$ is the desired edge. $\square$

2 Problems

1. We are given some number of balls in some some number of containers. Exactly k containers are empty. We rearrange these balls in such a way that after rearrangement none of the containers is empty. A ball is called *sad* if the number of balls in its current container is less that the number of balls in its previous container before rearrangement. Show that there are at least $k + 1$ sad balls. Is this bound tight?

2. There are A analysts and B algebraists taking part in the meeting of the faculty council. During the meeting every analyst got into a fight with at least one algebraist and every algebrais got into a fight with at most n analysts. It is also known that for every analyst, the number of his opponents-algebraists is strictly greater than the number of opponents-analysts of each of these opponents-algebraists. Show that $A \leq \frac{n}{n+1}B$.

3. Let k, n be positive integers. Each cell of an $n \times n$ table is filled with numbers 0 or 1. If some cell of the table contains 0, then the sum of the element in its cross (that is, in the union of cells lying with it in the same row or the same column) is at least $2k$. Find the least possible sum of numbers in the table.

4. Show that every simple planar graph with minimal vertex degree at least 3 contains an edge $\{x, y\}$ with $\deg(x) + \deg(y) \leq 13$. Is this bound tight?

5. Let C be a convex polyhedron with no quadrilateral and no pentagonal faces. Show that C has at least 4 triangular faces.

6. Suppose we are give a convex n-gon P on the plane and m red points distinct from the vertices of the polygon. Assume that each segment between two vertices of the polygon P contains at least one red point. Prove the inequality
$$m \geq n\left(1 + \frac{1}{2} + \ldots + \frac{1}{[(n-1)/2]}\right).$$

7. A square is cut into several triangles. Prove that there are two triangles sharing a common edge.

8. There are n lines in the plane, such that no three of them share a common point and no two of them are parallel. These lines split the plane into several parts. Prove that there are at least $n - 2$ triangles among them. Is this bound tight?