Introduction to Combinatorics
Analysis on the hypercube

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1. Find all functions $f : \{-1,1\}^n \to \mathbb{C}$ satisfying $f(x \cdot y) = f(x)f(y)$ for all $x, y \in \{-1,1\}^n$.

2. *In this problem we are going to go through combinatorial proof of the isoperimetric inequality on the discrete cube.*

   Let $u, v \in \{-1,1\}^n$. Let *canonical path* between u and v consist of vertices $p_0, p_1, \ldots, p_k \in \{-1,1\}^n$, where $p_0 = u, p_k = v$ which we get if we change differing bits in u and v (from u to v) from left to right.

   For example the canonical path between $(1,1,-1,1)$ and $(-1,1,1,-1)$ is

   $$(1,1,-1,1) \to (-1,1,-1,1) \to (-1,1,1,1) \to (-1,1,1,-1).$$

   Prove that $2^{n-1}|\partial A| \geq |A| \cdot |A^c|$ using properties of canonical paths.

3. Let $f : \{-1,1\}^n \to \{-1,1\}$ and consider the unique expansion $f = \sum_{S \subseteq [n]} a_S w_S$. The degree of f is defined as $\deg(f) = \max\{|S| : a_S \neq 0\}$. Show that

   $$\sum_{i=1}^{n} |a_{\{i\}}| \leq \deg(f).$$

4. Let $a_1, \ldots, a_n$ be real numbers such that $\sum_{i=1}^n a_i^2 = 1$. Show that

   $$\sum_{\varepsilon_1,\ldots,\varepsilon_n \in \{-1,1\}} \left| \sum_{i=1}^{n} a_i \varepsilon_i \right| \;\geq\; 2^{n-\frac{1}{2}}$$

   and show that this inequality is optimal.

   Hint. You may want to follow these steps:

   (a) For $f : \{-1,1\}^n \to \{-1,1\}$ define $(Lf)(x) = \frac{1}{2}\sum_{|y-x|=2}(f(y) - f(x))$. Prove that $\mathrm{Var}(f) \leq \mathbb{E}[f(-Lf)]$ by using Fourier analysis on the discrete cube.

   (b) Show that the above inequality improves to $\mathrm{Var}(f) \leq \frac{1}{2}\mathbb{E}[f(-Lf)]$ if f is an even function.

   (c) Let $f(x) = |\sum_{i=1}^n a_i x_i|$. Show that $-Lf \leq f$.

5. Let $A \subseteq \{-1,1\}^n$ be a monotone subset (that is, if $x_i \leq y_i$ for all $1 \leq i \leq n$ then $(x_1, \ldots, x_n) \in A$ implies $(y_1, \ldots, y_n) \in A$). Let n be odd and define

   $$\mathrm{Maj} = \left\{ (x_1, \ldots, x_n) \in \{-1,1\}^n : \sum_{i=1}^{n} x_i > 0 \right\}.$$

   Prove that $|\partial \mathrm{Maj}| \geq |\partial A|$.