Introduction to Combinatorics
Analysis on the hypercube

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1. Since $f(1) = f(1 \cdot 1) = f(1)^2$, we get $f(1) = 1$ or $f(1) = 0$. The latter leads to $f(x) = f(1)f(x) = 0$. The former gives $f(1) = f(x_1^2, \dots, x_n^2) = f(x)^2$ and thus $f(x) \in \{-1, 1\}$. We get that each $f(1, \dots, x_i, \dots, 1)$ is either identically 1 or is equal to x_i.

   Moreover
$$f(x_1, \dots, x_n) = \prod_{i=1}^{n} f(1, \dots, x_i, \dots, 1).$$

   Thus $f = w_S$ for some $S \subseteq [n]$ (namely, that set S consists of indices i such that $f(1, \dots, x_i, \dots, 1) = x_i$).

2. Let us calculate the number of canonical paths passing through a fixed directed edge.

$$(L0, x_i, R0) \to \ldots \to (L1, x_i, R0) \to (L1, -x_i, R0) \to \ldots \to (L1, x_i, R1).$$

To determine the a canonical path we have to choose $L0$ and $R1$. Therefore we have $2^{i-1} \cdot 2^{n-i} = 2^{n-1}$ canonical paths passing through the edge

$$(L1, x_i, R0) \to (L1, -x_i, R0).$$

Take $A \subseteq \{-1, 1\}^n$. There are $|A| \cdot |A^c|$ canonical paths $x \to \ldots \to y$, where $x \in A$ and $y \in A^c$. Each of them passes through some edge from ∂A and at most 2^{n-1} passes through a fixed edge. It follows that

$$|A| \cdot |A^c| \leq 2^{n-1} |\partial A|.$$

3. Let $\mathbb{E}$ denote the expectation with respect to the uniform distribution on the hypercube (as in the lectures). For $x = (x_1, \ldots, x_n)$ let $x^i = (x_1, \ldots, -x_i, \ldots, x_n)$. We have

$$|a_{\{i\}}| = |\mathbb{E}[f w_{\{i\}}]| = \frac{1}{2}\,|\mathbb{E}[f(x_1, \ldots, 1, \ldots x_n) - f(x_1, \ldots, -1, \ldots x_n)]| = \frac{1}{2}\mathbb{E}[|f(x) - f(x^i)|]$$
$$= \frac{1}{4}\mathbb{E}|f(x) - f(x^i)|^2 = \mathbb{E}[|\nabla_i f|^2] = \sum_{S:\ i \in S} a_S^2,$$

where the last equality follows from the fact that $\nabla_i f = \sum_S a_S \nabla_i w_S = \sum_{S: i \in S} a_S \nabla_i w_S$ and form Parseval's identity (this part has been showed during the lecture). Summing over i we get

$$\sum_{i=1}^n |a_{\{i\}}| \leq \sum_{i=1}^n \sum_{S:\ i \in S} a_S^2 = \sum_S |S| a_S^2 \leq \deg(f) \sum_S a_S^2 = \deg(f).$$

4. We shall work on the discrete cube. For $f : \{-1,1\}^n \to \{-1,1\}$ define $(Lf)(x) = \frac{1}{2}\sum_{y\sim x}(f(y)- f(x))$. We have $Lw_S = \frac{1}{2}(-|S|w_S + (n-|S|w_S)) - \frac{n}{2}w_S = -|S|w_S$. We claim that always

$$\mathrm{Var}(f) \le \mathbb{E}[f(-Lf)].$$

To see this observe that by using Parseval's identity the left hand side is equal to $\sum_{S\neq\emptyset} a_S^2$ whereas the right hand side is

$$\mathbb{E}[f(-Lf)] = \langle f, (-L)f\rangle = \left\langle \sum_S a_S w_S, \sum_S |S| a_S w_S \right\rangle = \sum_S |S| a_S^2$$

and the inequality follows.

We now claim that if f is even, that is $f(x) = f(-x)$ then

$$\mathrm{Var}(f) \le \frac{1}{2}\mathbb{E}[f(-Lf)].$$

Since f we have $a_S = 0$ whenever $2 \nmid |S|$ (since $a_S = \langle f, w_S\rangle = 0$ as $f w_S$ is odd), we get $f = \sum_{S:2||S|} a_S w_S$. The inequality reduces to

$$\sum_{S\neq\emptyset} a_S^2 \le \frac{1}{2}\sum_{S:2||S|} |S| a_S^2$$

and is clearly true.

We now define $f(x_1,\ldots,x_n) = |a_1x_1 + \ldots + a_nx_n|$ and observe that by the triangle inequality

$$\begin{aligned}(-Lf)(x) &= \frac{n}{2}f(x) - \frac{1}{2}\left(|-a_1x_1 + a_2x_2 + \ldots + a_nx_n| + \ldots + |a_1x_1 + a_2x_2 + \ldots - a_nx_n|\right) \\ &\le \frac{n}{2}f(x) - \frac{1}{2}(n-2)f(x) = f(x).\end{aligned}$$

Thus $-Lf \le f$. Since f is even and non-negative, we get

$$\mathrm{Var}(f) = \mathbb{E}[f^2] - (\mathbb{E}[f])^2 \le \frac{1}{2}\mathbb{E}[f(-Lf)] \le \frac{1}{2}\mathbb{E}[f(-Lf)] \le \frac{1}{2}\mathbb{E}[f^2].$$

In other words $\mathbb{E}[f^2] \le 2(\mathbb{E}[f])^2$. From the assumption $\sum_{i=1}^n a_i^2 = 1$ we get $\mathbb{E}[f^2] = 1$. Thus $\mathbb{E}[f] \ge 2^{-1/2}$, is precisely the desired inequality.

The equality holds for $a_1 = a_2 = \frac{1}{\sqrt{2}}$ and $a_3 = \ldots = a_n = 0$.

5. We are going to present three different solutions. Because of that we will be going into less details.

   Solution 1.

   For $u, v \in \{-1,1\}^n$ such that $u_i \leq v_i$ and $2 + \sum u_i = \sum v_i$ let us call u a *predecessor* of v and v a *successor* of u. If $s(u)$ is a number of ones in u then u has $s(u)$ predecessors and $n - s(u)$ successors. If there is an element a in A such that all its predecessors are not in A and such that $2s(a) < n$ then by removing it from A we will get a monotone set with a set whose boundary is bigger by $n - 2s(a)$. Similarly, if there is an element a in A^c whose all successors are in A then by adding it to A we increase boundary of A by $2s(a) - n$.

   Because of that if A is not Maj then we can increase its boundary what quickly proves the thesis.

   Solution 2.

   If an edge of hypercube goes from u to v such that u is a predecessor of v than there are exactly $s(u)!(n-1-s(u))!$ path shortest paths from $(-1,\ldots,-1)$ to $(1,\ldots,1)$ passing through it. Note that every such path passes through exactly one edge of boundary of A. Since $s(u)!(n-1-s(u))!$ is minimized when $s(u) = \frac{n-1}{2}$ and edges on boundary of Maj are exactly edges minimizing it, it can be argued that Maj has the biggest possible boundary out of monotone sets.

   Solution 3.

   By abuse of notation, let $A(x)$ be a function denoting whether $x \in A$ ($A(x) = 1$ if $x \in A$ and $A(x) = -1$ if $x \notin A$).

   Let's consider function $A(x)x_i$. We can see that this expression is directly proportional to the number of edges in boundary of A in the i-th direction. Hence $A(x)(x_1 + \ldots + x_n)$ is directly proportional to the size of boundary of A.

   We can see that $\mathbb{E}(A(x)(x_1 + \ldots + x_n)) \leq \mathbb{E}(|x_1 + \ldots + x_n|) = \mathbb{E}(Maj(x)(x_1 + \ldots + x_n))$, hence $|\partial Maj| \geq |\partial A|$.