Introduction to Combinatorics
Analysis on the hypercube

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1. You may start by inspecting what is $f(1, \ldots, 1)$.

   When dealing with completely multiplicative function on integers it is often good to know what are its values in arguments which are prime numbers (or powers of primes if we deal with a function that is multiplicative, but not completely multiplicative, but let us not dive into that detail). They are "building blocks" that are sufficient to determine its value in all integers. What would be the analogue of that on $\{-1, 1\}^n$.

2. How many canonical paths go through a particular directed edge of hypercube?

3. Observe that since f has values in the set $\{-1, 1\}$ we have $\frac{1}{2}|f(x) - f(x^i)| = \frac{1}{4}|f(x) - f(x^i)|^2$ where $x^i = (x_1, \ldots, -x_i, \ldots, x_n)$. The techniques used in the proof of Theorem 46 from the lecture notes may also be useful.

4. (a) How does L act on w_S?

   (b) How does the Fourier expansion of an even function look like?

   (c) Triangle inequality!

5. There are actually various ways to solve this problem, so here are three starting points for three different solutions:

   (a) Try changing A by one element (either add or remove one vertex, so that it stays monotone). How does $|\partial A|$ change?

   (b) Try considering all shortest $n!$ paths from $(-1, \ldots, -1)$ to $(1, \ldots, 1)$. How many such paths go through each edge?

   (c) Consider functions $g_i(x) = f(x)x_i$, where $f(x)$ is a function denoting whether $x \in A$.