Introduction to Combinatorics

Probabilistic method 2 – Problems

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1. We are given a fair coin and we toss it n times. Let X be the random variable denoting number of heads we got. Determine $\mathbb{E}X^3$.

2. Wojtek is playing Icy Tower. In this game you have a character that jumps on higher and higher platforms. In every jump, if a character is on i-th platform he makes a jump to one of platforms $i+1, i+2, i+3, i+4, i+5$, each with $\frac{1}{5}$ probability. Maximal contiguous sequence of jumps by two or more platforms is called a combo. If a combo started on platform i and ended on platform j then Wojtek gets $(j-i)^2$ points for that. His total score is 10 times the index of platform where the game ended plus scores for all combos. For example, if a game consisted of jumps with heights $3, 1, 2, 4$ then Wojtek gets $10(3+1+2+4)+3^2+(2+4)^2 = 145$ points. Wojtek got bored after n jumps and purposefully lost. Compute his expected score.

   Comment: The result may be not pretty, but the method itself is important here.

3. Let $p \in (\frac{1}{2}, 1)$. Consider an infinite rooted binary tree T. For each of its edges we delete it with probability $1-p$ and let R be the graph formed by edges that survived. Let C be a connected component of R containing root of T. Prove that C is infinite with positive probability.

4. Let $x_1, \ldots, x_n$ be boolean variables. A *literal* is a boolean variable x_i or its negation $\overline{x_i}$. A k-formula is an **AND** of clauses, each being an **OR** of k distinct literals. Such a formula ϕ is satisfiable if there exists an assignment $a \in \{0,1\}^n$ of values to variables for which $\phi(a) = 1$. We say that two clauses overlap if they have a common variable x_i, regardless of whether the variable is negated or not in the clauses.

   Let ϕ be a k-formula. Show that if each of its clauses overlaps with less than 2^{k-2} clauses, then ϕ is satisfiable.

5. Let $G = (V, E)$ be a graph with maximum degree not exceeding d. Let $V = V_1 \cup \ldots \cup V_r$ be a partition of V into r pairwise disjoint sets. Suppose that for $1 \leq i \leq r$ we have $|V_i| \geq 2ed$ (here e is the base of the natural logarithm). Prove that there is an independent set of vertices that contains a vertex from each set V_i.

6. Prove that $R(3,k) = \Omega(\frac{k^2}{\log^2 k})$, i.e. there exists a positive constant C such that for any positive integer k you can color edges of clique on $\lfloor C\frac{k^2}{\log^2 k} \rfloor$ vertices with red and blue such that there is no red triangle and no blue k-clique.