Introduction to Combinatorics
Probabilistic method 2 – Solutions

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1. In many combinatorial problems, random variables we are dealing with often express number of events that happened. In these cases we can express these random variables as sums of indicators of these events. Here, it is going to help us significantly since we will be able to expand this sum and focus on conjunctions of a constant number of events.

   Let A_i denote the event that we got heads in i-th toss. Then $X = \mathbb{1}_{A_1} + \ldots + \mathbb{1}_{A_n}$. Hence $\mathbb{E}X^3 = \mathbb{E}(\mathbb{1}_{A_1}+\ldots+\mathbb{1}_{A_n})^3 = \mathbb{E}(\sum_{1\le i\le n}\mathbb{1}_{A_i}^3 + 3\sum_{1\le i,j\le n, i\ne j}\mathbb{1}_{A_i}^2\mathbb{1}_{A_j} + 6\sum_{1\le i<j<k\le n}\mathbb{1}_{A_i}\mathbb{1}_{A_j}\mathbb{1}_{A_k})$. However we can use linearity of expectation and the fact that $\mathbb{1}_A^c = \mathbb{1}_A$ for any positive integer c and get that $\mathbb{E}X^3 = \sum_{1\le i\le n}\mathbb{E}\mathbb{1}_{A_i} + 6\sum_{1\le i<j\le n}\mathbb{E}\mathbb{1}_{A_i}\mathbb{1}_{A_j} + 6\sum_{1\le i<j<k\le n}\mathbb{E}\mathbb{1}_{A_i}\mathbb{1}_{A_j}\mathbb{1}_{A_k}$. Since all events are independent and hold with probability $\frac{1}{2}$ we get that $\mathbb{E}\mathbb{1}_{A_i} = \frac{1}{2}, \mathbb{E}\mathbb{1}_{A_i}\mathbb{1}_{A_j} = \frac{1}{4}$ and $\mathbb{E}\mathbb{1}_{A_i}\mathbb{1}_{A_j}\mathbb{1}_{A_k} = \frac{1}{8}$ if i, j, k are distinct values. Hence $\mathbb{E}X^3 = n \cdot \frac{1}{2} + 6\binom{n}{2}\frac{1}{4} + 6\binom{n}{3}\frac{1}{8}$.

2. *Solution 1.*

Let X_i be the random variable denoting height of i-th jump. Total score consists of two parts — first one is $10(X_1 + \ldots + X_n)$. It is very easy to compute expected value of this part, since $\mathbb{E}X_i = 3$ we get that $\mathbb{E}10(X_1 + \ldots + X_n) = 30n$. So, we can focus on part with combos.

Let $x_1, x_2, \ldots, x_n$ be an arbitrary sequence of jumps our character made. Then if jumps $x_l, \ldots, x_r$ form a combo then $(x_l + \ldots + x_r)^2$ is added to our score and we can expand it to $\sum_{l \le i \le r} x_i^2 + 2 \sum_{l \le i < j \le r} x_i x_j$. Hence, we can conclude that for some $i < j$ the term $x_i x_j$ is present in that sum expressing total values of combos if there is a combo containing both i-th jump and j-th jump, which happens if and only if all jumps with indices from interval $[i, j]$ were by two or more platforms. Term x_i^2 is present if and only if $x_i \ge 2$. Their total contribution is $n \cdot \frac{1}{5}(2^2 + 3^2 + 4^2 + 5^2) = \frac{54n}{5}$. Probability that term $x_i x_j$ is present is p^{j-i+1}, where $p = \frac{4}{5}$ denotes the probability that a particular jump is by at least two platforms. If we assume that it is present then its expected value is $\frac{(2+3+4+5)(2+3+4+5)}{4 \cdot 4} = \frac{49}{4}$. Hence it remains to compute sum of these probabilities for all pairs of $1 \le i < j \le n$ which is $S := \sum_{1 \le i < j \le n} p^{j-i+1}$. Then, our result will be $30n + \frac{54n}{5} + \frac{49}{2} S$.

We will compute S by grouping by $j - i$. There are $n - c$ such pairs with $j - i = c$. Hence $S = \sum_{c=1}^{n-1} (n-c) p^{c+1}$. After some long calculations it can be checked that $S = \frac{16}{5} n - 16 + 16 \cdot \left(\frac{4}{5}\right)^n$.

Therefore, final result is $30n + \frac{54n}{5} + \frac{49}{2} \cdot \left(\frac{16}{5} n - 16 + 16 \cdot \left(\frac{4}{5}\right)^n\right) = \frac{596}{5} n - 392 + 49 \cdot 2^{2n+3} \cdot 5^{-n}$.

Solution 2.

We can solve this problem inductively by determining some recurrences. Let S_i be the random variable denoting score after i jumps. However in order to determine difference between S_i and S_{i-1} we also need to keep track of current combo. Hence, let C_i be the random variable denoting height of current combo after i jumps (that is, the sum of the longest suffix consisting of jumps by two or more platforms). We can see that

$$\mathbb{E}S_i = \mathbb{E}S_{i-1} + 10 \cdot \frac{1+2+3+4+5}{5} + \frac{2^2+3^2+4^2+5^2}{5} + 2 \cdot \frac{2+3+4+5}{5} \cdot \mathbb{E}C_{i-1}$$

$$\mathbb{E}C_i = \frac{4}{5} \mathbb{E}C_{i-1} + \frac{2+3+4+5}{5}$$

These relations follow from similar arguments as in previous solution. If current combo is C_{i-1} and last jump is X_i and $X_i \ge 2$ then the difference in current combo score before and after last jump is $(C_{i-1} + X_i)^2 - C_{i-1}^2 = 2C_{i-1}X_i + X_i^2$. Expected contribution of term X_i^2 is $\frac{2^2+3^2+4^2+5^2}{5}$. Since C_{i-1} and X_i are independent variables we know that $\mathbb{E}C_{i-1}X_i = \mathbb{E}C_{i-1}\mathbb{E}X_i$. However remember that this happens only when $X_i \ge 2$.

Using these relations we can get the same result as in previous solution.

3. Let X_n be the random variable denoting number of vertices in C that are at distance n from the root. It can be easily seen that probability that C is infinite is $\lim_{n\to\infty} \mathbb{P}(X_n > 0)$. Because of that we would like to prove that there exists a positive constant c such that $\mathbb{P}(X_n > 0) > c$ for every n.

   We can simply determine $\mathbb{E}X_n$. There are 2^n vertices that are at distance n from the root and each of them belongs to C if whole path from it to root is in R and there is p^n probability for this, hence $\mathbb{E}X_n = (2p)^n$. Since $p > \frac{1}{2}$ we see that it goes to infinity as n goes to infinity, however that alone doesn't guarantee us a satisfying lower bound on $\mathbb{P}(X_n > 0)$, because a priori it could be the case that this expected value is an effect of some very rare events contributing a lot and vast majority of cases contributing zero to it. However, in such hypothetical case, variance of X_n would be very big. Because of that we may try to proceed in following two steps:

   1) Deduce some general lower bound on $\mathbb{P}(X > 0)$ depending on $\mathbb{E}X$ and $Var(X)$.
   2) Determine $Var(X_n)$ in order to apply derived bound.

   $\mathbb{P}(X > 0)$ can be expressed as an expected value of a random variable $\mathbb{1}_{X>0}$. Now, we can write $\mathbb{E}X \le \mathbb{E}(X \cdot \mathbb{1}_{X>0}) \le \sqrt{\mathbb{E}X^2 \cdot \mathbb{E}\mathbb{1}_{X>0}} = \sqrt{\mathbb{E}X^2\mathbb{P}(X > 0)}$, where the key inequality follows from Cauchy-Schwarz. Hence, if $\mathbb{E}X \ge 0$ we get that $\mathbb{P}(X > 0) \ge \frac{(\mathbb{E}X)^2}{\mathbb{E}X^2}$. It's not exactly variance here, as I may have suggested, but $\mathbb{E}X^2$ often conveys similar intuition.

   Let us now proceed to calculating $\mathbb{E}X_n^2$ in order to use our freshly derived bound. As you may have learnt in previous problems, it may be beneficial to express X_n as a sum of indicators of events it consists of. Let us call vertices at distance n from the root as $v_1, \ldots, v_{2^n}$. Let A_i be the event that $v_i \in C$. $X_n = \sum_{i=1}^{2^n} \mathbb{1}_{A_i}$, so $\mathbb{E}X_n^2 = \mathbb{E}(\sum_{i=1}^{2^n} \mathbb{1}_{A_i})^2 = \sum_{i=1}^{2^n}\sum_{j=1}^{2^n} \mathbb{E}\mathbb{1}_{A_i}\mathbb{1}_{A_j}$. However, $\mathbb{1}_{A_i}\mathbb{1}_{A_j} = \mathbb{1}_{A_i \cap A_j}$. Event $A_i \cap A_j$ is an event where both v_i and v_j belong to C and that happens if and only if sum of their paths to the root is in R. This happens with probability $p^{2n-k(i,j)}$, where $k(i,j)$ is the number of edges in the intersection of paths from v_i and v_j to root. One can see that for a fixed i there are exactly 2^{c-1} vertices j such that $k(i,j) = n - c$ for $c > 0$ and 1 vertex (itself) for $c = 0$. Hence

   $$\mathbb{E}X_n^2 = 2^n(p^n + p^{n+1} + 2p^{n+2} + \ldots + 2^{n-1}p^{2n}) =$$

   $$2^np^n + 2^np^{n+1}(1 + 2p + \ldots + (2p)^{n-1}) = 2^np^n + 2^np^{n+1}\frac{(2p)^n - 1}{2p - 1}.$$

   Now we can compute that

   $$\mathbb{P}(X_n > 0) \ge \frac{(\mathbb{E}X_n)^2}{\mathbb{E}X_n^2} = \frac{2^{2n}p^{2n}}{2^np^n + 2^np^{n+1}\frac{(2p)^n-1}{2p-1}} = (2p-1)\frac{2^{2n}p^{2n}}{2^np^n + 2^{2n}p^{2n+1} - 2^np^{n+1}} \ge$$

   $$\ge (2p-1)\frac{2^{2n}p^{2n}}{2^{2n}p^{2n} + 2^{2n}p^{2n+1}} = \frac{2p-1}{1+p}.$$

   $\frac{2p-1}{1+p}$ is indeed a positive constant, what completes our proof.

4. Consider a random assignment of values to variables x_i (each is 0 or 1 with probability 1/2). Let A_i be the event that the i-th clause of ϕ is not satisfied. If i-th clause contains literals x and $\overline{x}$ for some variable x, then it is always satisfied and $\mathbb{P}(A_i) = 0$. Otherwise it contains literals using k distinct variables. For this clause to be not satisfied all literals of this clause have to receive wrong values, thus $\mathbb{P}(A_i) = 2^{-k}$. Clearly A_i is mutually independent of the family of events A_j such that the corresponding clauses C_j do not overlap with the clause C_i. Thus we can use the LLL with $d = 2^{k-2} - 1$. Since $e2^{-k}2^{k-2} = e/4 < 1$ we have $\mathbb{P}\left(\bigcap_i A_i^c\right) > 0$, which means that there exists a good assignment.

5. We can assume that for each i one has $|V_i| = k_0$ where $k_0 = \lceil 2ed \rceil$. Otherwise we just replace V_i with its subset V_i' of cardinality k_0 and consider an induced subgraph on $V_1' \cup \ldots \cup V_r'$. We now pick from each V_i a vertex x_i uniformly and independently at random (each element will be chosen with probability $1/k_0$). Let $S = \{x_1, \ldots, x_r\}$ be the resulting random set of vertices. For each edge $f \in E$, let A_f be the event that S contains both ends of f. Then clearly $\mathbb{P}(A_f) \leq \frac{1}{k_0^2}$. Let's denote $p = \frac{1}{k_0^2}$. Moreover, if the endpoints of f lie in V_i and V_j then the event A_f is mutually independents of the family of events $\{A_g : g = \{u, v\}, u, v \notin V_i \cup V_j\}$. Thus the family of events $\{A_f : f \in E\}$ has a dependency graph of maximal degree less than $D = 2k_0 d - 2$ (the number of edges going out of $V_i \cup V_j$, excluded the edge f). Note that $ep(D+1) \leq e\frac{1}{k_0^2} 2k_0 d = 2ed/k_0 \leq 1$. LLL finishes the proof.

6. Let us take a clique on n vertices and color its edges randomly in red and blue keeping fingers crossed we get a coloring such that there is no red triangle and blue k-clique. Since triangles are much smaller objects than general k-cliques, we should color each edge in red with probability $p := p(k)$ that should be significantly smaller than $\frac{1}{2}$ (and, obviously, choices for different edges are independent).

   There are $\binom{n}{3}$ events of a red triangle showing up (call them *red events*) and $\binom{n}{l}$ events of a blue k-clique showing up (call them *blue events*). In our first attempt we could just count expected number of these events that happen and hope it is smaller than 1 for appropriately chosen p, but if you do the calculations you will see it doesn't give satisfying result. However, we can see that very small fraction of events concerning red triangles are dependent to each other. If we take triangles T_1 and T_2 such that they do not share an edge then their corresponding events are independent. A triangle can share an edge with at most $3n$ other triangles, which is much smaller number than their total number. Moreover, each k-clique can share an edge with at most $\binom{k}{2}n$ triangles. We may want to take an advantage of these facts in Lovasz Local Lemma. However, red events and blue events are highly asymmetric and that is why we may want to use its general asymmetric version.

   For each event we are going to associate a real number from $(0, 1)$ interval with it. Since we have two types of events it seems like a good idea to associate the same value for all red events (call it x) and the same value for all blue events (call it y), but x should probably be different than y. Since each red event is dependent to at most $3n$ other red events and at most $\binom{n}{l}$ blue events and each blue event is dependent to at most $\binom{k}{2}n$ red events and at most $\binom{n}{l}$ blue events in order to apply Lovasz Local Lemma we should have

$$p^3 = \mathbb{P}(R_i) \leq x(1-x)^{3n}(1-y)^{\binom{n}{l}}$$

$$(1-p)^{\binom{k}{2}} = \mathbb{P}(B_i) \leq y(1-x)^{\binom{k}{2}n}(1-y)^{\binom{n}{l}}$$

   where R_i and B_i are arbitrary red and blue events.

   So, if for a particular value of n we prove that there exist real numbers p, x, y from $(0, 1)$ satisfying these inequalities, based on Lovasz Local Lemma we can conclude that $R(3, k) \geq n$. Now, this problem turns to a less pleasant analytical one... It turns out that such numbers can be found for $n = \Omega(\frac{k^2}{\log^2 k})$, however since calculations are a bit lengthy, let me omit them, but if you want to follow them then you can find them in this link from page 6.