Introduction to Combinatorics
Probabilistic method – Problems

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1. Let G be a graph with n vertices and m edges. Prove that we can partition $V(G)$ into A and B such that there are at least $\frac{m}{2}$ edges with one endpoint in A and one in B.
2. Let $k \geq 2$ be an integer. Prove that we can color integers from 1 to $\lfloor\sqrt{2^k(k-1)}\rfloor$ in two colors, so that there are no k integers with the same color forming an arithmetic sequence.
3. Let G be a graph with n vertices and m edges and let $d = \frac{2m}{n}$ be its average degree. Prove that G has:
   (a) an independent set of size at least $\frac{n}{2d}$
   (b) an independent set of size at least $\frac{n}{d+1}$.
4. Let $A_1, \ldots, A_n, B_1, \ldots, B_n$ be finite sets of integer numbers such that for every $1 \leq i \leq n$ it holds that $A_i \cap B_i = \emptyset$ and for every $1 \leq i < j \leq n$ it holds that $(A_i \cap B_j) \cup (A_j \cap B_i) \neq \emptyset$. Prove that for every $x \in [0, 1]$ it holds that $\sum_{i=1}^{n} x^{|A_i|}(1-x)^{|B_i|} \leq 1$.
5. We are given set of n lines in general position that cuts the plane into some regions. We call a subset A of them *good* if there is no region with finite area (which is not cut by any other line) whose all sides are parts of lines from A. Prove that there exists a good set of lines:
   (a) of size at least $\frac{\sqrt{n}}{2}$
   (b) of size at least $\sqrt{n}$

   Definition: A set of lines in the plane is in general position if no two are parallel and no three pass through the same point.

   Comment+Hint: (b) may be very hard. You may try using discharging for it. Probabilistic method probably will not be enough.