Introduction to Combinatorics
Problems

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1. Show that a graph with minimum degree $\delta \geq 2$ has a cycle of length at least $\delta + 1$.
2. For every integer $n \geq 3$ determine the biggest number $f(n)$ so that there exists a graph on n vertices with $f(n)$ edges which doesn't have a Hamiltonian cycle.
3. Let G be an n-vertex graph with degrees $d_1 \leq d_2 \leq \ldots \leq d_n$. Prove that if $d_k \geq k+1$ for $k < \frac{n}{2}$, then G contains a Hamiltonian cycle.
4. On an $n \times n$ chessboard there are $2n$ pawns. Prove that there is a nonempty subset A of them so that in every row and in every column there is an even number of pawns from A.

   Hint: Try to express this problem in terms of bipartite graphs. (Bipartite graph is a graph whose vertex set is partitioned into two sets A and B and every edge has one endpoint in A and second one in B).
5. There are n disks $D_1, \ldots, D_n$ with disjoint interiors drawn on the plane. Prove that there exists $1 \leq i \leq n$ such that D_i is tangent to at most 5 other disks.

   Bonus: Find all values of k so that there exists a drawing of disks $D_1, \ldots, D_n$ so that for each D_i there are exactly k other disks which are tangent to it.
6. Let G be a planar graph and $H_1, \ldots, H_k$ be its subgraphs with following properties:

   - each H_i is connected
   - they are all pairwise disjoint, i.e. $i \neq j \Rightarrow H_i \cap H_j = \emptyset$
   - for every pair of H_i and H_j (for $i \neq j$) there exist vertices $v_i \in H_i$ and $v_j \in H_j$ such that $v_i v_j \in E(G)$

   Prove that $k \leq 4$.