Introduction to Combinatorics
Graphs 1 – Solutions

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1. Let's greedily build a long path. Start with some vertex v_1. If v_i has a neighbour that has not appeared yet in $v_1, \ldots, v_{i-1}$ then set it as v_{i+1} (if there is more than one than choose any of them). If it turns out that there is no such vertex then it means that all neighbours of v_i are within set $v_1, \ldots, v_{i-1}$. Since there are at least δ of them, there is $j \leq i - \delta$ such that v_j is a neighbour of v_i. Then $v_j v_{j+1} \ldots v_i$ forms a cycle and its length is $j - i + 1 \geq \delta + 1$. (Instead of described procedure we could have started with the longest path since it has no neighbours which are not already on this path. However, if you are a programmer you may prefer greedy approach since it can be turned into a linear time algorithm for finding such cycle!)

2. Consider a graph which is a clique on $n-1$ vertices with added a vertex with a single neighbour in that clique. It has n vertices, $\binom{n-1}{2}+1$ edges and doesn't have a Hamiltonian cycle, so we know that $f(n) \geq \binom{n-1}{2}+1$. We will prove that if a graph on n vertices has more than $\binom{n-1}{2}+1$ edges then it contains Hamiltonian cycle what will prove that $f(n) = \binom{n-1}{2}+1$.

   Let's try to prove by induction that if a graph has at least $g(n) = \binom{n-1}{2}+2$ edges than it contains Hamiltonian cycle. Base of induction ($n=3$) is trivial. Assume that it is true for $1, 2, \ldots, n-1$ and try to prove it for n. Let's take some vertex v with degree $d(v)$. If $g(n-1)+d(v) \leq g(n)$ then we know that after deleting it we can find Hamiltonian cycle on the remaining $n-1$ vertices. If we find two vertices on that cycle which are adjacent on it and which are neighbours of v then we can put v between these two on that cycle creating Hamiltonian cycle on the whole graph. However one can simply observe that $d(v) > \frac{n-1}{2}$ is a sufficient condition for having two neighbours which are adjacent on that Hamiltonian cycle on $n-1$ vertices.

   Because of that, we can make this argument work if $d(v)$ satisfies two conditions:

   - $g(n-1)+d(v) \leq g(n) \Leftrightarrow d(v) \leq n-2$
   - $d(v) > \frac{n-1}{2}$

   Therefore, if there is any vertex v in G such that $\frac{n}{2} \leq d(v) \leq n-2$ then we know that our graph contains Hamiltonian cycle.

   Let us now consider remaining cases where there are only vertices with degrees exactly $n-1$ or at most $\frac{n-1}{2}$. Let us denote by c the number of vertices with degree $n-1$. Number of edges in a graph is a half of sum of degrees which is at most $\frac{1}{2}((n-1)c + \frac{n-1}{2}(n-c))$. Moreover vertices with degree $n-1$ are adjacent to all other vertices, so if there is any vertex with degree at most $\frac{n-1}{2}$ then $c \leq \frac{n-1}{2}$ (if there are no such vertices then our graph is a clique which is trivial case). However we assumed that there are at least $g(n) = \binom{n-1}{2}+2$ edges in our graph. Direct calculation shows that for $n \geq 4$ we have $g(n) > \frac{1}{2}((n-1)c + \frac{n-1}{2}(n-c))$ which proves that such case leads to a contradiction.

3. By contrary assume that our graph is not Hamiltonian. Let us use the trick from Ore's Theorem proof and add edges as long as it is possible to add one while not creating a Hamiltonian cycle. Of course, the degree condition still holds after adding some edges (even though the order of degrees may have been changed), so we can assume that adding any edge to our graph creates Hamiltonian cycle.

   Let us number vertices $v_1, \ldots, v_n$ so that vertex v_k has degree d_k. Let us take a pair of vertices v_k and v_l which are non-neighbours so that $k + l$ is maximal possible and $k < l$. We know that adding edge between them creates a Hamiltonian cycle, so there is already a Hamiltonian path between them. If $d_k + d_l \geq n$ then by mimicing proof of Ore's Theorem we get that there exists a Hamiltonian cycle in that graph. If $k \leq \frac{n}{2}$ then we know that $d_k \geq \frac{n}{2}$ and since $d_l \geq d_k$ then $d_l \geq \frac{n}{2}$, so $d_k + d_l \geq n$. If $k < \frac{n}{2}$ then $d_k \geq k + 1$. Since $k + l$ is maximal we know that v_l is adjacent to all vertices $v_{k+1}, \ldots, v_n$, except itself, which is $n - k - 1$ vertices. Hence $d_k + d_l \geq (k + 1) + (n - k - 1) = n$, so we are done.

4. Let us think about this problem in a following way. We have some pawns, and putting every pawn allows us to change the parity of number of pawns in some particular row and in some particular column. We can imagine $2n$ counters – n for columns and n for rows and every pawn tells us "you can use me to increase counter for i-th row and j-th column". Because of that it is a natural idea to consider a bipartite graph on $n+n$ vertices where we put an edge between i-th vertex on the left and j-th vertex on the right if and only if there is a pawn in i-th row and j-th column. Now, after translating our statement to a language of that bipartite graph, we would like to find a nonempty set of edges so that every vertex has even degree. However since there are $2n$ pawns, there are $2n$ edges in that graph as well which is exactly as many as vertices. Therefore this graph contains a cycle. If we choose edges of that cycle only, we know that every vertex has even degree (since it would be either 0 or 2), so we are done.

5. *Solution 1.*: Let us put a vertex in the center of every disk. Let's draw an edge between two such vertices if their corresponding disks touch. It is clear that graph that we will get is planar. And every planar graph has a vertex of degree at most 5 which means that there is a disk which is tangent to at most 5 other disks.

   Solution 2.: There is in fact a simple solution that doesn't use theory of planar graphs. Assume that every disk is tangent to at least 6 other disks. Let us take circle with the minimum radius. If there is more than one then let us take one with the lowest value of its y coordinate. Simple geometry shows that the only case when it is possible is if these 6 circles have the same radius as well and their centers form a regular hexagon whose center is center of our initial circle (for proof of that you can use a fact that the biggest angle in a triangle lies opposite to its longest side and since angles in triangle sum to 180 degrees, it has to be at least 60 degrees). However one of these circles has smaller value of its y coordinate, hence we got a contradiction.

   Bonus: For the bonus part, using original question, we can conclude that $k \leq 5$. It now suffices to find drawings of disks for $k = 0, 1, 2, 3, 4, 5$. For $k \leq 2$ it is trivial. For $k = 3, 4, 5$ if you are determined enough, you will find them as well, however proving its existence for $k = 5$ "by hand" may be not that easy. There are two other ways that we can follow.

   First one uses stereographic projections. Let us take tetrahedron for $k = 3$, cube for $k = 4$ and dodecahedron for $k = 5$. Let us take one particular solid out of the mentioned ones and inscribe circles in every face of it. It is clear that whatever solid we take, every circle is tangent to exactly k other circles. Moreover there exists a sphere that contains all of these circles (i.e. these circles are sections of that sphere). Each such circle determines a "cap" on that sphere (smaller of the two parts of the sphere that this circle divides it to). Now let us take some point which is outside of these caps and use it to project all these circles onto some plane. Magic of stereographic projection causes that projection of a circle on a sphere is a circle on that plane as well. Moreover, since this point was outside all caps, projections of these circles have disjoint interiors and it is still the case that every circle is tangent to k other circles, so we are done (if we had projected using point within a cap, there would have been one big circle whose interior would contain all other circles which is not what we wanted).

   Second way uses Circle Packing Theorem which characterizes exactly possible graphs that we can get as graphs of tangencies of disks with disjoint interiors. We already know that all such graphs are trivially planar. However, the reverse implication is surprisingly true as well which is the statement of Circle Packing Theorem. In fact, for every planar graph G, we can find disks whose tangencies graph is exactly G. Because of that it suffices to check whether for every $0 \leq k \leq 5$ there exists a k-regular planar graph. However we can easily check that all of them exist, for $k = 3$ it is an adjacency graph of a tetrahedron, for $k = 4$ it is an adjacency graph of octahedron and for $k = 5$ it is an adjacency graph of icosahedron (and we know that all of them are planar).

6. In this problem we are given connected disjoint subgraphs of a planar graph so that each pair of them is connected by at least one edge. It kind of feels like a clique where every vertex is blown up to some bigger region. So, intuitively it should behave in the same way, meaning that it is not possible to have such “clique” on five regions, right? But how do we make this intuition formal?

   Let us introduce an operation called *edge contraction*. Contraction takes two vertices u and v such that uv is an edge. Let $N(u)$ and $N(v)$ denote their neighbourhoods. Operation of contracting edge uv deletes vertices u and v and puts a vertex w into graph G whose neighbourhood is $(N(u) \cup N(v)) \setminus \{u, v\}$. Depending on a specific setting, sometimes one may want keep many edges to the same vertex if it was neighbour of both u and v, but since we are working in simple graphs, we use it in a way which was described.

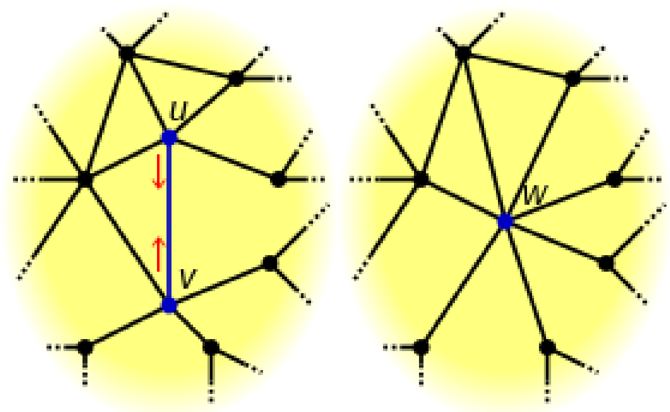

Figure 1: Contraction of an edge uv to vertex w

   It is very intuitive that contraction of an edge preserves planarity of a graph, but we do not give a formal proof here. Instead of contracting just an edge we may go one step further and contract whole connected subgraph. We do that by consecutively contracting edges of a spanning tree of these connected subgraphs (one can see that order of contractions doesn't matter). Contraction of connected subgraph, basically replaces it with a single vertex which has edges to all vertices v so that there was a vertex u within that subgraph such that edge uv existed originally. We may now contract subgraphs $H_1, \ldots, H_k$ to single vertices $h_1, \ldots, h_k$ and since edge contraction preserves planarity we know that what we are left with is a planar graph as well. However vertices $h_1, \ldots, h_k$ form a clique, so since K_5 is not planar we conclude that $k \leq 4$.

   Comment 1: If a graph H can be obtained from G by sequence of vertex deletions and edge deletions then H is simply called a subgraph of G. However if we allow edge contractions as well then we say that H is a *minor* of G. What we did in this exercise was to prove that minors of planar graphs are planar hence planar graphs cannot have K_5 as minor.

   Comment 2: On the lecture we saw operation of putting a vertex of degree 2 in the middle of an edge. Such operation is called *edge subdivision* and if H can be obtained from G by sequence of edge subdivision then G is called a *topological minor* of H. One can see that by contracting arbitrary out of two edges coming out of such vertex we return to the state before edge subdivision, so if A is a topological minor of B then A is a minor of B as well. However reverse implication doesn't have to be true.