Introduction to Combinatorics
Problems

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1. Let p be a prime, n a positive integer, and L a subset of $\{0, 1, \ldots, p-1\}$. Let $\mathcal{F}$ be a family of subsets of $[n]$ such that for any $A, B \in \mathcal{F}$ where $A \neq B$ we have $|A \cap B| \mod p \in L$ and $|A| \mod p \notin L$. Prove that $|\mathcal{F}| \leq \sum_{k=0}^{|L|} \binom{n}{k}$.

2. Let n be a positive integer and L a subset of integers. Let $\mathcal{F}$ be a family of subsets of $[n]$ such that for any $A, B \in \mathcal{F}$ where $A \neq B$ we have $|A \cap B| \in L$. Prove that $|\mathcal{F}| \leq \sum_{k=0}^{|L|} \binom{n}{k}$.

3. Let $\mathbb{F}_q$ be a field on q elements and P be a polynomial in n variables $x_1, \ldots, x_n$ of degree at most $q-1$ over $\mathbb{F}_q$. Prove that if P vanishes on whole $\mathbb{F}_q^n$ then all its coefficients are zero.

4. Let n, d be positive integers. What is the dimension of the linear space of polynomials in n variables of degree at most d?

5. Let n be a positive integer and $\mathbb{F}_q$ be a field on q elements. For $x, v \in \mathbb{F}_q^n$ we call a set $\{x + tv : t \in \mathbb{F}_q\}$ a *line*. Let B be a subset of $\mathbb{F}_q^n$ such that for every $x \in \mathbb{F}_q^n \setminus B$ there is a line L such that $L \setminus B = \{x\}$. Prove that $|B| \geq \binom{n+q-2}{n}$.

   Hint: How to prove that $|A| \geq m$ for some set A? You can find an injective linear mapping from linear space over $\mathbb{F}_q$ with dimension m to $\mathbb{F}_q^A$ ($\mathbb{F}_q^A$ is the linear space of functions from A to $\mathbb{F}_q$).

6. In $\mathbb{R}^3$ some n points are coloured. In every step, if four coloured points lie on the same line, we can colour any other point on this line. It turns out that we can colour any point $P \in \mathbb{R}^3$ in a finite number of steps (possibly depending on P). Find the minimal value of n for which this could happen.

   Hint: Think about some polynomials.