Introduction to Combinatorics
Problems

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1. We basically repeat what was said on the lecture and slightly adjust it to this problem. For a set $A \subseteq [n]$ we define function $f_A : \{0,1\}^n \to \mathbb{Z}_p$ in the following way: $f_A(x_1, \ldots, x_n) = \prod_{k \in L}(\sum_{i \in A} x_i - k)$. We note that $f_A(\mathbb{1}_B) = \prod_{k \in L}(|A \cap B| - k)$. Hence if $A \neq B \Rightarrow |A \cap B| \in L$ we deduce that $A \neq B \Rightarrow f_A(\mathbb{1}_B) = 0$ and since $|A| \notin L$ we deduce that $f_A(\mathbb{1}_A) \neq 0$. Hence if we treat these functions as elements of linear space we can conclude that they are linearly independent. However they are spanned by functions $x_{i_1} x_{i_2} \ldots x_{i_j}$ (recall that even though f takes values in $\mathbb{Z}_p$ we in fact have $x_i \in \{0,1\}$, so $x_i^c = x_i$ for positive c), where $j \leq |L|$ and $1 \leq i_1 < i_2 < \ldots < i_j \leq n$ and number of such functions is at most $d = \binom{n}{0} + \binom{n}{1} + \ldots + \binom{n}{|L|}$ what proves that dimension of space that contains all functions f_A is at most d, so number of them is at most d since they are linearly independent.

2. We try following the idea that was used in previous problem, but we need to do some twists. In our first attempt we may try do define $f_A(x_1, \ldots, x_n) = \prod_{k \in L}(\sum_{i \in A} x_i - k)$, but the problem with it is that it could be the case that $|A| \in L$ what will cause $f_A(\mathbb{1}_A) = 0$. In order to prevent this we may remove term for $k = |A|$ if $|A| \in L$ from that product. Moreover since $|A \cap B| \le |A|$ terms with $k > |A|$ actually do not change whether f_A is zero or not, so for convenience we may remove them as well and in our second attempt define $g_A(x_1, \ldots, x_n) = \prod_{k \in L, k < |A|}(\sum_{i \in A} x_i - k)$. It follows that $g_A(\mathbb{1}_A)$ is nonzero, however it is not necessarily the case that if $A \neq B$ then $g_A(B) = 0$ since it could be the case that $|A \cap B| = |A|$. Under wishful assumption that there is no such a pair of sets in $\mathcal{F}$ such that one contains the other we would actually be done since we would be able to conclude functions g_A are linearly independent based on the fact that for $A, B \in \mathcal{F}$ we have $g_A(\mathbb{1}_B) \neq 0 \Leftrightarrow A = B$ and continue in the same way we did in the previous problem. We cannot assume that but in fact it turns out we are still able to argue that they are linearly independent!

   Let us order elements of $\mathcal{F}$ in nondescending order of their sizes. $|A_1| \le |A_2| \le \ldots \le |A_{|\mathcal{F}|}|$. We can now note that if $i < j$ then it implies $g_{A_j}(\mathbb{1}_{A_i}) = 0$ because if $|A_i| \le |A_j|$ and $A_i \neq A_j$ then we have that $|A_i \cap A_j| < |A_j|$, so there exists a term in product g_{A_j} which will be equal to zero. Because of that if we create a $|\mathcal{F}| \times |\mathcal{F}|$ matrix M where $m_{ij} = g_{A_i}(A_j)$ then this matrix is upper-triangular with nonzero entries on the main diagonal. Such matrix is of course non-singular what proves that functions g_{A_i} are linearly independent and we conclude the same way we did before.

3. By grouping the monomials in P by the power x_n has we can say that there exist polynomials on $n-1$ variables P_0, P_1, …, P_{q-1} such that $P(x_1, \ldots, x_{n-1}, x_n) = x_n^0 P_0(x_1, \ldots, x_{n-1}) + x_n^1 P_1(x_1, \ldots, x_{n-1}) + \ldots + x_n^{q-1} P_{q-1}(x_1, \ldots, x_{n-1})$. If we now fix variables $x_1, \ldots, x_{n-1}$ we can treat right hand side as a polynomial in one variable — namely x_n. We can write $Q_{x_1,\ldots,x_{n-1}}(x) = \sum_{i=0}^{q-1} x^i P_i(x_1, \ldots, x_{n-1})$. Since P vanishes for all points we know that Q has q roots which is more than its degree. It means that all its coefficients are zero. It means that we have $P_i(x_1, \ldots, x_{n-1}) = 0$ for all i and for all choices of $x_1, \ldots, x_{n-1}$. It means that all polynomials $P_0, \ldots, P_{q-1}$ vanish on all their arguments spaces. Therefore we can apply induction here (base case is trivial) and conclude that all coefficients of P are zero.

4. Dimension of said space is number of different monomials $x_1^{c_1} \dots x_n^{c_n}$ where $c_1, \dots, c_n$ are nonnegative integers summing of to at most d, so we need to count those.

   There is a clear bijection between sequences of nonnegative integers of length n summing up to at most d and sequences of nonnegative integers of length $n+1$ summing up to exactly d. In order to create this bijection we just append difference of d and sum $c_1 + \dots + c_n$ to sequence $c_1, \dots, c_n$ to get the sequence of length $n+1$ and sum being exactly d.

   So now, our question in how many sequence of nonnegative integers of length $n+1$ are there. This is well-known problem of putting d indistinguishable balls into $n+1$ distinguishable buckets (sometimes called “stars and bars” problem as well) and the answer to it is $\binom{n+d}{n}$. That is because we can think about this problem as a problem of ordering d balls and n “walls” between buckets.

5. $\binom{n+q-2}{n}$ is the dimension of space of all polynomials on n variables of degree at most $q-2$. Based on that and the hint next to the problem statement we should find linear map from that space to the space of function from B to $\mathbb{F}_q^n$. What is that map? It is quite clear — just the evaluation, i.e. we map polynomial P into a function from B to $\mathbb{F}_q$ which at every point of A takes value which P takes there. In order to prove that this mapping is injective we need to prove that its kernel is trivial i.e. if some polynomial P from our space vanishes on whole set B it in fact is zero polynomial, i.e. all its coefficients are zero.

   In fact based on third problem it suffices to show that P vanishes on whole $\mathbb{F}_q^n$ and we are going to do exactly that. We know that P vanishes on B (that is our assumption), so let us take some point x outside of B and prove that it vanishes there as well. We know that there is a line containing x so that x is the only point outside of B on that line. Let direction of that line be v, so it is a set of points of form $x+tv$ where $t \in \mathbb{F}_q$. For fixed x, v we can express $P(x+tv)$ as a polynomials in one variable — namely t. We know that this polynomial has degree at most $q-2$, but it has $q-1$ roots on it, so in fact it is a zero polynomial, so it vanishes in x as well, so we are done.

6. It is easy to construct a set of 20 starting colored point that suffice to color every point of $\mathbb{R}^3$ in finite number of steps. Take for example set of points (x, y, z) such that $x, y, z \in \mathbb{Z}_{\geq 0}$ and $x + y + z \leq 3$.

   Now we will prove that 20 is optimal. Crucial fact that will lead us to the solution is that if cubic polynomial has at least four zeros it has to be zero everywhere. Assume that P is some polynomial in variables x, y, z of degree at most 3. Let us restrict P to a line in $\mathbb{R}^3$. Restricting to line means considering points $x = x_0 + td_x, y = y_0 + td_y, z = z_0 + td_z$ for some fixed real numbers $x_0, y_0, z_0, d_x, d_y, d_z$ and t being any real number, where (x_0, y_0, z_0) is one of points on this line and (d_x, d_y, d_z) is its direction. After such substitution it becomes a polynomial in one variable (namely t) of degree at most 3 which means that if it has four zeros on that line it has to be zero everywhere on that line. Therefore it can be proven that if P attains zero at some set of colored point, all points that we can color in finite number of steps need to be roots of P as well. However space of polynomials in three variables of degree at most 3 has in fact dimension 20, so if we start our construction with 19 points there exists a polynomial whose values are zeros in all these points but which is nonzero what means there are points where it has nonzero values which in turn cannot be colored in finite number of steps.

   By the way, if you haven't solved this problem - don't worry. When it was posed on an international competition nobody solved it. But at least you got some hints and preceding problems to help you along the way :).