Introduction to Combinatorics

Wojciech Nadara, class II, 05/03/2020

1. Let $v_1, \ldots, v_n$ be real numbers such that $|v_i| \geq 1$ for $i = 1, \ldots, n$. Define

$$A = \{x = (x_1, \ldots, x_n) \in \{-1, 1\}^n : \quad |v_1 x_1 + \cdots + v_n x_n| < 1\}.$$

   Prove that $|A| \leq \binom{n}{[n/2]}$.

   In other words, the probability that the n step random walk with steps $\pm v_i$ (each taken with probability $\frac{1}{2}$) ends up in the interval $[-1, 1]$ is upper bounded by $2^{-n}\binom{n}{[n/2]} = O(1/\sqrt{n})$.

2. Suppose that in a given finite partial order the maximal lenght of a chain is equal to r. Prove that this partial order can be partitioned into r antichains.

3. Let s, r be positive integers. Show that in any partial order on a set of $n \geq sr + 1$ elements, there exists a chain of length $s + 1$ or an antichain of size $r + 1$.

4. Let s, r be positive integers. Show that every sequence of $sr + 1$ real numbers contains a non-decreasing subsequence of length $s + 1$ or a non-increasing subsequence of length $r + 1$.

5. Show that the above theorem is tight.

6. Find $R(3, 3)$ and $R(4, 3)$.

7. Show that for any integer $m \geq 3$ there exists an integer $n = n(m)$ such that any set of n points in the Euclidean plane, no three of which are collinear, contains m points which are the vertices of a convex m-gon.

8. Show that for every $r \geq 2$ there exists $n = n(r)$ such that in every coloring of $\{1, \ldots, n\}$ with r colors there exist monochromatic triple of distinct numbers satisfying $x + y = z$.