Introduction to Combinatorics

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1. Let $A_1, \ldots, A_m$ be distinct subsets of an n element set. Suppose that $A_i \cap A_j \neq \emptyset$ for all i, j. Show that $m \leq 2^{n-1}$.

2. Prove that if $\mathcal{F}$ is a family of distinct pairwise intersecting subsets of an n element set X, then there exists a family $\mathcal{F}'$ of distinct pairwise intersecting subsets of X, such that $\mathcal{F} \subseteq \mathcal{F}'$ and $|\mathcal{F}'| = 2^{n-1}$.

3. Let $A_1, \ldots, A_m$ be a family of distinct subsets of an n element set, such that $|A_i|$ and $|A_i \cap A_j|$ are even for all i, j. Prove that $m \leq 2^{[n/2]}$. Is this bound tight?

4. Let n be odd. Let $A_1, \ldots, A_m$ be a family of distinct subsets of an n element set, such that $|A_i|$ is even for all i and $|A_i \cap A_j|$ is odd for all i, j. Prove that $m \leq n$. Is this bound tight?

5. Let A be a $2n \times 2n$ matrix with zeroes on the main diagonal and ± 1 elsewhere. Show that A is non-singular over $\mathbb{R}$.

6. Suppose $\mathbb{F}$ is a subfield of $\mathbb{G}$. Suppose $v_1, \ldots, v_k$ are linearly independent in the vector space $(\mathbb{F}^n, \mathbb{F})$. Does it follow that $v_1, \ldots, v_k$ are linearly independent in the vector space $(\mathbb{G}^n, \mathbb{G})$?

7. A family $S_1, \ldots, S_k$ of subsets of a given set X is called a *sunflower* with k petals and core A (it could be that $A = \emptyset$) if $S_i \cap S_j = A$ for all $i \neq j$ and $S_i \setminus A$ is nonempty for all i.

   Prove that every family of s element subsets of X satisfying $|\mathcal{F}| > s!(k-1)^s$ contains a sunflower with k petals.

8. Let $\mathcal{F}$ be an antichain of subsets (with an inclusion order) of an n element set. Suppose that all of these sets have cardinality at most k where $2k \leq n$. Show that $|\mathcal{F}| \leq \binom{n}{k}$.

9. Let $n \leq 2k$ and let $A_1, \ldots, A_m$ be distinct k element subsets of a give set X with n elements. Suppose $A_i \cup A_j \neq X$ for all i, j. Show that $m \leq (1 - \frac{k}{n})\binom{n}{k}$.

10. Let $A_1, \ldots, A_m$ and $B_1, \ldots, B_m$ be subsets of a given finite set X such that $A_i \cap B_j = \emptyset$ if and only if $i = j$. Let $a_i = |A_i|$ and $b_i = |B_i|$. Prove the inequality

$$\sum_{i=1}^{m} \binom{a_i + b_i}{a_i}^{-1} \leq 1.$$

11. Let $A_1, \ldots, A_m$ be a element subsets and $B_1, \ldots, B_m$ be b element subsets of a given finite set X, such that $A_i \cap B_j = \emptyset$ if and only if $i = j$. Show that $m \leq \binom{a+b}{a}$. Is this bound tight?

12. Let $v_1, \ldots, v_n$ be real numbers such that $|v_i| \geq 1$ for $i = 1, \ldots, n$. Define

$$A = \{x = (x_1, \ldots, x_n) \in \{-1, 1\}^n : \quad |v_1 x_1 + \cdots + v_n x_n| < 1\}.$$

   Prove that $|A| \leq \binom{n}{[n/2]}$.

   In other words, the probability that the n step random walk with steps $\pm v_i$ (each taken with probability $\frac{1}{2}$) ends up in the interval $[-1, 1]$ is upper bounded by $2^{-n}\binom{n}{[n/2]} = O(1/\sqrt{n})$.