Introduction to combinatorics

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1 Graphs

1.1 Networks

In this section we are going to investigate flow networks. They are imagined as networks of pipes with some capacities where some amount of water flows from source to sink through these pipes.

We introduce the following definitions and notation.

- A pair (V, E), where V is a finite set and $E \subseteq V \times V$, is called a **directed graph**. An element $(x, y) \in E$ will be called a **directed edge** from x to y.
- A triple (V, E, c) is called a **network** if (V, E) is a directed graph and $c : V \times V \to [0, \infty)$ is some non-negative function on pairs of vertices. If (x, y) is an edge then $c(x, y)$ will be called the **capacity** of the edge (x, y). If there is no (x, y) edge then $c(x, y) = 0$.
- Let (V, E, c) be a network and $s, t \in V$ where s will ba called a *source* and t a *sink* (or *target*). For a function $f : V \times V \to \mathbb{R}$ and $v \in V$ we define $x_f(v) = \sum_{y \in V} f(v, y)$ called **excess** of v. A function $f : V \times V \to \mathbb{R}$ is called an $s - t$ **flow** if

  (i) $f(u, v) \leq c(u, v)$ for every $x, y \in V$,

  (ii) $f(u, v) = -f(v, u)$ (k units of flow from u to v can be imagined as $-k$ units of flow from v to u)

  (iii) for every $v \notin \{s, t\}$ we have $x_f(v) = 0$ (Kirchhoff's law, influx to a vertex is equal to outflux from it),

  The quantity $x_f(s)$ will be called the **value** of f and will be denoted by $|f|$. It is easy to prove that under Kirchhoff's law we have $x_f(s) = -x_f(t)$.
- For a flow f and subsets $A, B \subseteq V$ we define $x_f(A, B) = \sum_{a \in A, b \in B} f(a, b)$.
- An $s - t$ flow f is called **maximum** if $|f| = \max_{f'} |f'|$ where the maximum is taking over all $s - t$ flows. Note that the maximum is attained due to an easy compactness argument.
- For a network (V, E, c) and a pair of vertices $s, t \in V$ such that there is a path from s to t, an $s - t$ **cut** is a subset $\Pi \subseteq E$ such that there is no path from s to t if we remove Π from E. The **capacity** of the cut Π is $C(\Pi) = \sum_{e \in \Pi} c(e)$. The cut is called **minimum** if it has the minimum value of $C(\Pi)$ among all $s - t$ cuts.

We are ready to prove the first easy lemma.

Lemma 1. Let (V, E, c) be a network and let f be a flow from s to t. Suppose $A \subseteq V$ is such that $s \in A$ and $t \notin A$. Then $x_f(A, V \setminus A) = |f|$.

Proof. Since $x_f(v) = 0$ for $v \neq s, t$, we have

$$|f| = x_f(s) = \sum_{v \in A} x_f(v) = x_f(A, V) = x_f(A, A) + x_f(A, V \setminus A) = x_f(A, V \setminus A)$$

□

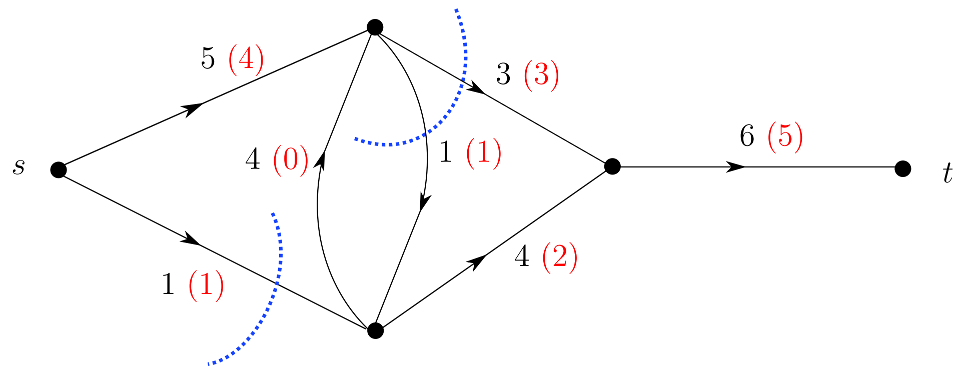

Figure 1: A network with source s, and sink t. Black numbers indicate edge capacities, red numbers indicate values of a maximum flow. In blue – minimum cut.

We can now show that the value of any flow is upper bounded by the capacity of any cut.

Lemma 2. Let (V, E, c) be a connected network with source s and sink t. Then for any flow f and for any $s - t$ cut Π we have $|f| \le C(\Pi)$.

Proof. Suppose that we are given a flow f and a cut Π separating s from t. Let A be the set of reachable vertices from s after deleting from the network all the edges belonging to Π (which does not contain t). Then from Lemma 1 we get

$$|f| = x_f(A) = f(A, V \setminus A) = \sum_{(a,b)\in A\times(V\setminus A)} f(a,b) \le \sum_{(a,b)\in A\times(V\setminus A)} c(a,b) \le \sum_{(a,b)\in\Pi} c(a,b) = C(\Pi),$$

where the last but one inequality follows from the fact that every edge (a, b) from A to $V \setminus A$ must belong to the cut (otherwise b would be in A). □

The above lemma shows that $\max_f |f| \le \min_\Pi C(\Pi)$. The celebrated Max-flow min-cut theorem of Ford and Fulkerson shows that this is in fact equality.

Theorem 1 (Ford-Fulkerson, 1962)**.** Let (V, E, c) be a connected network with source s and sink t. Then $\max_f |f| = \min_\Pi C(\Pi)$.

Proof. To this end, for a flow f, let us define a *residual network*. Let us define $c'(u, v) = c(u, v) - f(u, v)$, which intuitively denotes number of units of flow that we are able to push from u to v in addition to what is already pushed there. It is easy to check that residual network $R_f = (V, E', c')$ for a flow f will have following property: a function f' is a valid flow in residual network if and only if $f + f'$ is a valid flow in original network, since $f'(x, y) \le c'(x, y) \Leftrightarrow f(x, y) + f'(x, y) \le c(x, y)$.

Note that it could be the case that in original network there was an edge (u, v), but there was no edge (v, u), and we pushed some units of flow through that edge. In that case $c'(v, u) = c(v, u) - f(v, u) = 0 + f(u, v) = f(u, v)$, so we are able to push $f(u, v)$ units of flow in residual network in the direction from v to u in order to "cancel out" flow units that we pushed in original network. Hence, residual network can introduce some edges with nonzero capacities that were not present in original network.

Let f be the maximum flow (it exists due to an easy compactness argument). From Lemma 2 it suffices to show that there exists Π such that $|f| = C(\Pi)$. Let $R_f = (V, E', c')$ be a residual network for f.

Claim. R_f does not contain any $s-t$ path consisting of edges with positive residual capacities.

Proof. Assume by contrary that such path $(p_0, \ldots, p_k)$ exists, where $p_0 = s, p_k = t$.
Let $m = min(c'(p_0, p_1), c'(p_1, p_2), \ldots, c'(p_{k-1}, p_k))$, i.e. minimum residual capacity on this path. By assumption we have $m > 0$. Create a flow f' such that $f'(p_i, p_{i+1}) = m$ for $0 \leq i \leq k-1$ and $f'(u, v) = 0$ for all other pairs of vertices. We can easily see that f' is a valid residual flow of positive value, so $f + f'$ is a valid flow in original network. We have $|f + f'| - |f| = |f'| = m > 0$ what contradicts the assumption that f was maximum $s-t$ flow. □

Since R_f does not contain any $s-t$ path consisting of edges with positive residual capacities, there is a set A such that $s \in A, t \notin A$ and there are no edges with positive residual capacities from A to $V \setminus A$. It means that for every u, v such that $u \in A, v \notin A$ we have that $c'(u, v) = 0 \Leftrightarrow c(u, v) = f(u, v)$, so if Π is a set of edges from A to $V \setminus A$, then we have $|f| = x_f(A) = f(A, V \setminus A) = \sum_{u \in A, v \notin A} f(u, v) = \sum_{u \in A, v \notin A} c(u, v) = C(\Pi)$, hence $\max_f |f| \geq \min_\Pi C(\Pi)$. However, we already know that $\max_f |f| \leq \min_\Pi C(\Pi)$, so indeed it is the case that $\max_f |f| = \min_\Pi C(\Pi)$.

□

Remark. If all the capacities are integers, then from the above proof one gets that there exists a maximum flow with $f(e) \in \mathbb{Z}$ for any $e \in E$. Indeed the above argument works if we consider only integer-valued flows (note that in this case we have $m \in \mathbb{Z}$).