Introduction to Combinatorics

Combinatorics of convex sets

Wojciech Nadara, class 12, 2020-05-28

1. Let $\mathcal{F}$ be a finite family of convex sets in $\mathbb{R}^d$ and assume that $|\mathcal{F}| \geq d+1$. Let C be a convex subset of $\mathbb{R}^d$. Show that the following conditions are equivalent:

   (i) There exists a translate of C intersecting all the members of the family $\mathcal{F}$.

   (ii) For every subfamily $\mathcal{F}_0 \subseteq \mathcal{F}$ with $|\mathcal{F}_0| = d+1$ there exists a translate of C intersecting all the members of $\mathcal{F}_0$.

2. Let X be an n-element set in $\mathbb{R}^d$. A point x in $\mathbb{R}^d$ is called a *centrepoint* of X if every closed subspace containing x contains at least $\frac{n}{d+1}$ points of X.

   Show that every finite set in $\mathbb{R}^d$ has a centrepoint.

3. Let X and Y be finite sets of points in $\mathbb{R}^d$. Show that the following conditions are equivalent:

   (a) X and Y can be strictly separated by a hyperplane,

   (b) for every $Z \subseteq X \cup Y$ with $|Z| \leq d+2$ the sets $Z \cap X$ and $Z \cap Y$ can be strictly separated by a hyperplane.

   (A hypeplane H strictly separates sets A and B if A lies in one open halfspace determined by H and B lies in the opposite open halfspace.)

4. We say that points $x_1, \ldots, x_n$ in $\mathbb{R}^d$ are in a *convex position* if the set $\{x_1, \ldots, x_n\}$ is contained in the boundary of $\operatorname{conv}(\{x_1, \ldots, x_n\})$.

   Show that for every fixed positive integers k, d there exists an integer $N = N(k, d)$ with the following property: in every set of N distinct points in $\mathbb{R}^d$, such that no $d+1$ of them lie on a hyperplane, one can find k points in a convex position.

5. Let $d \geq 2$ and let K be a compact convex set in $\mathbb{R}^d$. Let $K^t = K \cap \{(x_1, \ldots, x_d) \in \mathbb{R}^d : x_1 = t\}$. Define the function $f : \mathbb{R} \to \mathbb{R}$ by $f(t) = \operatorname{vol}_{d-1}(K^t)$. Prove that $f^{\frac{1}{d-1}}$ is concave on its support.

   Remark. Here we treat the sets K^t as subsets of $\mathbb{R}^{d-1}$ and take their $(d-1)$-dimensional volume.