Introduction to Combinatorics
Combinatorics of convex sets

Wojciech Nadara, class 12, 2020-05-28

1. Let $t(K) = \{x \in \mathbb{R}^d : \ (x + C) \cap K \neq \emptyset\}$. It is enough to show that the sets $t(K)$ are convex and then apply Helly's theorem.

   To show that $t(K)$ is convex suppose that $(x + C) \cap K \neq \emptyset$ and $(y + C) \cap K \neq \emptyset$. We are to show that $(\lambda x + (1 - \lambda)y + C) \cap K \neq \emptyset$ for every $\lambda \in [0, 1]$. Suppose that $x_0 \in (x + C) \cap K$ and $y_0 \in (y + C) \cap K$. We are going to show that $\lambda x_0 + (1 - \lambda)y_0 \in (\lambda x + (1 - \lambda)y + C) \cap K$. To check it we observe that:

   (a) since $x_0, y_0 \in K$ by convexity of K we have $\lambda x_0 + (1 - \lambda)y_0 \in K$.

   (b) since $x_0 \in x + C$ and $y_0 \in y + C$ then $\lambda x_0 + (1 - \lambda)y_0 \in \lambda(x + C) + (1 - \lambda)(y + C) = (\lambda x + (1 - \lambda)y) + \lambda C + (1 - \lambda)C = (\lambda x + (1 - \lambda)y) + C$. The last equality $\lambda C + (1 - \lambda)C$ is just the convexity of C (for general sets only the inclusion $C \subseteq \lambda C + (1 - \lambda)C$ is true!).

2. This is the so-called Rado's theorem. It is straightforward to show that the following conditions are equivalent:

   (i) x is a centrepoint of X.

   (ii) x is in every open half-space H containing more than $\frac{dn}{d+1}$ points of X.

   We shall find x satisfying (ii). Consider the family of convex sets

$$\mathcal{C} = \left\{ \mathrm{conv}(X \cap H) : \ H \text{ is an open half-space such that } X \cap H > \frac{dn}{d+1} \right\}.$$

   This is clearly a finite family of size at most $2^{|X|}$. The intersection of any $d+1$ sets from $\mathcal{C}$ misses less that $(d+1)\frac{n}{d+1} = n$ points and thus it is non-empty. By Helly's theorem there exists $x \in \bigcap_{C \in \mathcal{C}} C$. Every such x is a desired point.

3. This is called the Kirchberger's theorem. For every $x \in X$ and every $y \in Y$ we consider halfspaces in $\mathbb{R}^{d+1}$ (these are **parameters** of halfspaces in $\mathbb{R}^d$)

$$G_x = \{(u, u_{d+1}) \in \mathbb{R}^{d+1} : \langle u, x \rangle > u_{d+1}\}, \qquad G_y = \{(u, u_{d+1}) \in \mathbb{R}^{d+1} : \langle u, y \rangle < u_{d+1}\}.$$

Note that if, say $(u, u_{d+1}) \in G_x \cap G_y$ then the halfspace $\{x \in \mathbb{R}^d : \langle u, x \rangle > u_{d+1}\}$ strictly separates x from y. Our assumption shows that every d+2 halfspaces have a point in common. By Helly's theorem all the halfspaces have a point in common, which determines a strictly separating halfspace.

4. Consider all $(d+1)$-element subsets of a finite set X of point in $\mathbb{R}^d$ and color them using the following rule: the set S gets the color $\chi(S) = n(S) \bmod d$, where $n(S)$ is the number of points from X in the interior of S. From Ramsey theorem if $|X|$ is sufficiently big, there will be k element subset A of X such that all its $(d+1)$-subsets received the same color χ. We claim that this is the desired subset.

   Indeed, suppose by contradiction that there is a point in $a \in A$ such that $a \in \operatorname{int}(\operatorname{conv}(A))$. From Caratheodory's theorem a is a convex combination of $d+1$ points $x_1, \ldots, x_{d+1}$ from A. Let $V = \{x_1, \ldots, x_{d+1}\}$ Let $K = \operatorname{conv}(\{x_1, \ldots, x_{d+1}\})$. We must have $a \in \operatorname{int}(K)$ since otherwise a would be on a face of the symplex K which implies that a together with this face forms a $(d+1)$-element coplanar set. For every face F_i $(i = 1, \ldots, d+1)$ of K consider the simplex $S_i = \operatorname{conv}(F_i, a)$ and the set $V_i \subseteq A$ of its vertices. We have, modulo d, and using the fact that none of the points from $A \cap \operatorname{int}(V)$ lies on $\bigcup_{i=1}^{n+1} \partial S_i$ (otherwise contradiction with non-complanarity)

$$\chi = \chi(V) = 1 + \sum_{i=1}^{d+1} \chi(V_i) = 1 + (d+1)\chi = \chi + 1,$$

   contradiction.

5. Assume that t, s are in the support of f, that is K_t and K_s are non-empty. We claim that for every $\lambda \in [0,1]$ we have

$$K_{\lambda t+(1-\lambda)s} \supseteq \lambda K_t + (1-\lambda)K_s.$$

Take any points $(t,a) \in K_t$ and $(s,b) \in K_s$. Here $a, b \in \mathbb{R}^{d-1}$. Clearly by convexity of K, we have

$$\lambda(t,a) + (1-\lambda)(s,b) = (\lambda t + (1-\lambda)s, \lambda a + (1-\lambda)b) \in K$$

and since the first coordinate of this vector is $\lambda t + (1-\lambda)s$, the above point actually belongs to $K_{\lambda t+(1-\lambda)s}$. Take $\tilde{K}_t = K_t - (t,0) \subseteq \mathbb{R}^{d-1} \times \{0\}$. Then $\tilde{K}_{\lambda t+(1-\lambda)s} \supseteq \lambda \tilde{K}_t + (1-\lambda)\tilde{K}_s$. From Brunn-Minkowski inequality we have,

$$\begin{aligned} f(\lambda t + (1-\lambda)s)^{\frac{1}{d-1}} &= \mathrm{vol}_{d-1}(\tilde{K}_{\lambda t+(1-\lambda)s})^{\frac{1}{d-1}} \geq \mathrm{vol}_{d-1}(\lambda \tilde{K}_t + (1-\lambda)\tilde{K}_s)^{\frac{1}{d-1}} \\ &\geq \mathrm{vol}_{d-1}(\lambda \tilde{K}_t)^{\frac{1}{d-1}} + (\mathrm{vol}_{d-1}((1-\lambda)\tilde{K}_s)^{\frac{1}{d-1}} \\ &= \lambda \mathrm{vol}_{d-1}(\tilde{K}_t)^{\frac{1}{d-1}} + (1-\lambda)\mathrm{vol}_{d-1}(\tilde{K}_s)^{\frac{1}{d-1}} \\ &= \lambda f(t)^{\frac{1}{d-1}} + (1-\lambda)f(s)^{\frac{1}{d-1}}, \end{aligned}$$

as desired. Note that in order to apply the Brunn-Minkowski inequality we have used the assumption that K_t and K_s are non-empty.