Introduction to Combinatorics
Generating functions

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1. a) Let $s_1, \ldots, s_{2n+1} \in \{-1, 1\}$ be such that $\sum_{i=1}^{2n+1} s_i = 1$. Let

$$\sigma_k = (s_k, \ldots, s_{2n+1}, s_1, \ldots, s_{k-1}).$$

   Show that all the sequences $\sigma_1, \ldots, \sigma_{2n+1}$ are different and precisely one of them has all its partial sums strictly positive.

   b) Use the above fact to show that the nth Catalan number equals $\frac{1}{n+1}\binom{2n}{n}$.

2. Let $S_n^{(132)}$ be the set of permutations $a_1 a_2 \ldots a_n$ of $\{1, 2, \ldots, n\}$ such that there are no indexes $i < j < k$ such that $a_i < a_k < a_j$. Find the cardinality of $S_n^{(132)}$.

3. Let $S_0 = 0, S_1, \ldots, S_n$ be a trajectory of a symmetric ($p = \frac{1}{2}$) random walk starting from the origin. Let $M_n = \max_{0 \le t \le n} S_t$. Prove that

   a) for all $h \ge 1$ and all k we have

$$\mathbb{P}\left(M_n \ge h, S_n = k\right) = \begin{cases} \mathbb{P}\left(S_n = k\right) & k \ge h \\ \mathbb{P}\left(S_n = 2h - k\right) & k < h \end{cases},$$

   b) for all $h \ge 1$ we have $\mathbb{P}\left(M_n \ge h\right) = \mathbb{P}\left(S_n = h\right) + 2\mathbb{P}\left(S_n \ge h + 1\right)$,

   c) for all $h \ge 1$ we have $\mathbb{P}\left(M_n = h\right) = \mathbb{P}\left(S_n = h\right) + \mathbb{P}\left(S_n = h + 1\right)$.

4. Let R_n be the number of ways to write n as a sum of pairwise different non-negative integers. Let S_n be the number of ways to write n as a sum of odd integers (permutation of numbers gives the same division). Using generating functions show that $R_n = S_n$.

   Bonus: Can you show that combinatorially as well?

5. Peter likes magnifying glasses. He has n dollars and decided he will spend them all to buy exactly k magnifying glasses. However magnifying glasses come in different prices and magnifying ratios (abbreviated as MR) – magnifying glass that costs d dollars has MR equal to $\binom{d+1}{2}$ (and there are infinitely many such glasses in stock for every positive integer d). He has not decided yet on how he will exactly spend his money, so he considers a scenario that he will spend $d_1, \ldots, d_k$ dollars for them respectively for each sequence of positive integers such that $d_1 + \ldots d_k = n$. After buying them, he will stack them up and their MRs will multiply, so if they have MRs $r_1, \ldots, r_k$, resulting construction will have MR $r_1 \cdot r_2 \cdot \ldots \cdot r_k$.

   a) Compute the sum of all MRs over all possible scenarios.

   b) Compute the sum of all MRs over all possible scenarios if magnifying glass that costs d dollars has magnifying ratio d^2 instead of $\binom{d+1}{2}$ (it is acceptable to express answer as a sum of length k).

Bonus: Can you solve a) combinatorially?