Introduction to Combinatorics
Generating functions

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1. (a) We consider the graph of the sequence of partial sums of our sequence (s_i) (see below). Consider the last moment k when the graph attains its lowest level $-h$. Consider the shift that starts from this moment, that is $s_{k+1}, \ldots, s_{2n+1}, s_1, \ldots, s_k$. From our construction we have

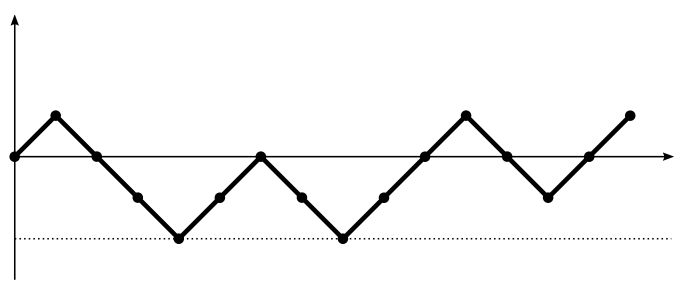
$$s_{k+1} \geq 1, \quad s_{k+1} + s_{k+2} \geq 1, \quad \ldots, \quad s_{k+1} + \ldots + s_{2n+1} = h + 1.$$

Since

$$s_1 \geq -h, \quad s_1 + s_2 \geq -h, \quad \ldots \quad s_1 + s_2 + \ldots + s_k \geq -h$$

we get

$$s_{k+1} + \ldots + s_{2n+1} + s_1 \geq 1, \quad \ldots \quad , s_{k+1} + \ldots + s_{2n+1} + s_1 + s_2 + \ldots + s_k \geq 1.$$



To show that every shift of (s_i) is different just observe that equality of two shifts implies that the sequence (s_i) is periodic, which cannot be the case as we have n values -1 and $n+1$ values $+1$.

(b) Note that C_n is the number of sequences $s_1, \ldots, s_{2n+1} \in \{-1, 1\}$ having all partial sums strictly positive and satisfying $\sum_{i=1}^{2n+1} s_i = 1$ (to get bijection with Dyck paths just remove the first step). We call such sequences ballot sequences. Not consider all sequences $a_1, \ldots, a_{2n+1}$ with $\sum_{i=1}^{2n+1} a_i = 1$ and no restriction on partial sums. There are $\binom{2n+1}{n}$ such sequences. We call two such sequences equivalent if one is obtained from the other by shifting. From point (a) we have $\frac{1}{2n+1}\binom{2n+1}{n}$ equivalence classes since each equivalence class has precisely $2n+1$ members. Again by point (a) in every equivalence class there is precisely one ballot sequence.

2. Let π be an element of $S_n^{(132)}$. We write π as $\pi = \pi_L n \pi_R$. Any number in π_L is bigger than any number in π_R. Thus π_R is a permutation of $\{1, 2, \ldots, |\pi_R|\}$ and π_L is a permutation of $\{|\pi_R| + 1, \ldots, n - 1\}$. Let π'_L be the permutation of $\{1, \ldots, |\pi_L|\}$ obtained from π_L by substracting $|\pi_R|$ from each of its letter. We define the map f between $S_n^{(132)}$ and Dyck paths of length $2n$ in the following way

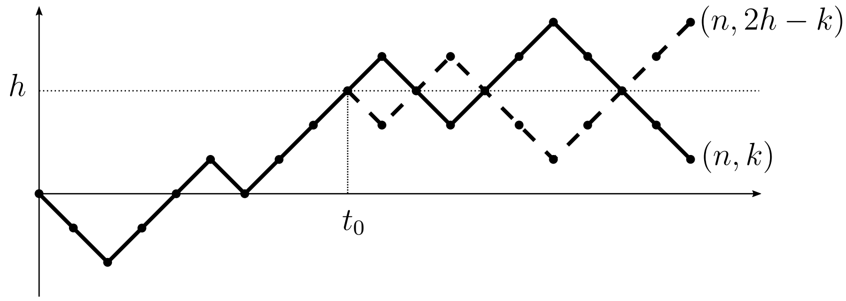
$$f(\pi) \;=\; \nearrow f(\pi'_L) \;\searrow f(\pi_R), \qquad f(\emptyset) = \emptyset.$$

For example:

$$\begin{aligned} f(7564213) &= \nearrow\searrow f(564213) = \nearrow\searrow\nearrow f(1) \searrow f(4213) = \nearrow\searrow\nearrow\nearrow\searrow\searrow f(4213) \\ &= \nearrow\searrow\nearrow\nearrow\searrow\searrow\nearrow\searrow f(213) = \nearrow\searrow\nearrow\nearrow\searrow\searrow\nearrow\searrow\nearrow f(21) \searrow \\ &= \nearrow\searrow\nearrow\nearrow\searrow\searrow\nearrow\searrow\nearrow\nearrow\searrow f(1) \searrow = \nearrow\searrow\nearrow\nearrow\searrow\searrow\nearrow\searrow\nearrow\nearrow\searrow\nearrow\searrow\searrow \end{aligned}$$

It is not hard to describe the inverse map. We live it to the reader.

3. (a) For $k \geq h$ the assertion is trivial. We therefore assume $k < h$. Let t_0 be the first time when the trajectory reaches level h. Reflecting the trajectory for times $t_0 \leq t \leq n$ gives the desired equality.



(b) We have

$$\mathbb{P}(M_n \geq h) = \sum_k \mathbb{P}(M_n \geq h, S_n = k) = \sum_{k \geq h} \mathbb{P}(S_n = k) + \sum_{k<h} \mathbb{P}(S_n = 2h - k)$$
$$= \mathbb{P}(S_n = h) + 2 \sum_{k \geq h+1} \mathbb{P}(S_n = k) = \mathbb{P}(S_n = h) + 2\mathbb{P}(S_n \geq h+1).$$

(c) We have

$$\begin{aligned}
\mathbb{P}(M_n = h) &= \mathbb{P}(M_n \geq h) - \mathbb{P}(M_n \geq h+1) \\
&= \mathbb{P}(S_n = h) + 2\mathbb{P}(S_n \geq h+1) - \mathbb{P}(S_n = h+1) + 2\mathbb{P}(S_n \geq h+2) \\
&= \mathbb{P}(S_n = h) + 2(\mathbb{P}(S_n \geq h+1) - \mathbb{P}(S_n \geq h+2)) - \mathbb{P}(S_n = h+1) \\
&= \mathbb{P}(S_n = h) + 2\mathbb{P}(S_n = h+1) - \mathbb{P}(S_n = h+1) = \mathbb{P}(S_n = h) + \mathbb{P}(S_n = h+1).
\end{aligned}$$

4. Let $R(x)$ be the generating function representing the number of ways in which we can write an integer as a sum of pairwise different numbers, that is $R(x) = \sum_{i=0}^{\infty} R_i x^i$. It is clear that $R(x) = (1+x)(1+x^2)(1+x^3)\ldots$.

   Let $S(x)$ be the generating function representing the number of ways in which we can write an integer as a sum of odd numbers, that is $S(x) = \sum_{i=0}^{\infty} S_i x^i$. It is clear that $S(x) = (1+x+x^2+\ldots)(1+x^3+x^6+\ldots)(1+x^5+x^{10}+\ldots)\ldots = \frac{1}{1-x} \cdot \frac{1}{1-x^3} \cdot \frac{1}{1-x^5} \cdot \ldots$.

   Now, what is left to prove is that $S(x) = R(x)$. However note that $\frac{1}{1-x} = \prod_{i=0}^{\infty}(1+x^{2^i})$, and similarly $\frac{1}{1-x^k} = \prod_{i=0}^{\infty}(1+x^{k2^i})$. That is because if we multiply both sides by $1-x^k$ then left side will become 1 and right side will become $(1-x^k)(1+x^k)(1+x^{2k})(1+x^{4k})\ldots = (1-x^{2k})(1+x^{2k})(1+x^{4k})\ldots = (1-x^{4k})(1+x^{4k})\ldots$ which is equal to 1 (well, kinda...). If we multiply such equalities for all odd values of k then we get that $S(x) = R(x)$, hence $S_n = R_n$.

   Comment: All operations on these infinite products may look very shaky. However, at their heart, they are in fact counting finite objects, hence intuitively no weird problems with limits should be a real obstacle here. A good way to formalize this is to look at these equalities modulo x^n for some particular n. Then all these products become finite and all coefficients next to x^i for $i < n$ stay unchanged.

   Bonus:

   This problem can be solved as well in combinatorial way, but it may be tricky to spot, whereas solution from generating functions may be more natural. However this solution in fact gives basically a whole insight on how combinatorial solution should look like. We may skip shortening $1+x+x^2+\ldots$ as $\frac{1}{1-x}$ and directly note that $1+x+x^2+\ldots = (1+x)(1+x^2)(1+x^4)\ldots$ since every natural number has a unique binary expansion. Such equality which is a side effect in solution using generating functions hints us that using numbers $c, 2c, 4c, 8c, \ldots$ at most once for odd c gives the same effect as using c any number of times, what quickly leads to fully combinatorial solution showing bijection between both types of expressions that changes k occurences of odd c to powers of two making up binary expansion of k multiplied by c and the other way around.

5. a) Summing over all positive integer sequences summing up to some specified value should like like something where generating functions are of great help, since

$$[x^n](c_0x^0 + c_1x^1 + \ldots)^k = \sum_{\substack{i_1,\ldots,i_k \geq 0 \\ i_1+\ldots+i_k=n}} c_{i_1}c_{i_2}\ldots c_{i_k}$$

($[x^n]F(x)$ stands for a coefficient next to x^n in $F(x)$).

A small difference is that in that formula we sum over nonnegative integers and we want to sum over positive integers and we will get that if we drop c_0x^0 term (what corresponds to a fact that there are no magnifying glasses costing 0 dollars).

Let $F(x)$ be a generating function for the sequence $\binom{d+1}{2}$ for $d = 0, 1, \ldots$, that is $F(x) = x\binom{2}{2} + x^2\binom{3}{2} + \ldots$. We know that $\frac{1}{(1-x)^c} = \sum_{i=0}^{\infty} x^i \binom{i+c-1}{c-1}$ for positive integers c, so we conclude that $F(x) = \frac{x}{(1-x)^3}$. Based on the formula we mentioned on the beginning, we know that $F(x)^k$ will be the generating function for the sequence of answers for all possible values of n.

$$F(x)^k = \frac{x^k}{(1-x)^{3k}} = \sum_{i=0}^{\infty} x^{i+k}\binom{i+3k-1}{3k-1} = \sum_{i=k}^{\infty} x^i \binom{i+2k-1}{3k-1}$$

Therefore, if Peter has n dollars then the magnifying ratios summed over all possible scenarios will be $\binom{n+2k-1}{3k-1}$.

b) We already know what we can do with $\binom{d+1}{2}$, so can we use that knowledge to cope with d^2? It seems so, as $d^2 = \binom{d+1}{2} + \binom{d}{2}$. Let us create analogous generating function:

$$F(x) = \sum_{i=0}^{\infty} x^i i^2 = \sum_{i=0}^{\infty} x^i \left(\binom{i+1}{2} + \binom{i}{2}\right) = \frac{x+x^2}{(1-x)^3}$$

We continue:

$$F(x)^k = \left(\frac{x+x^2}{(1-x)^3}\right)^k = \frac{x^k(\binom{k}{0}x^0 + \binom{k}{1}x^1 + \ldots + x^k\binom{k}{k})}{(1-x)^{3k}} =$$

$$= \sum_{a=0}^{k} \binom{k}{a} \sum_{i=0}^{\infty} x^{i+k+a} \binom{i+3k-1}{3k-1} = \sum_{a=0}^{k} \binom{k}{a} \sum_{i=0}^{\infty} x^i \binom{i+2k-1-a}{3k-1}$$

Hence, answer for n dollars equals $\sum_{a=0}^{k} \binom{k}{a}\binom{n+2k-1-a}{3k-1}$.

Note: You may want to express d^2 in a bit different way, for example $d^2 = 2\binom{d+1}{2} - d$ or $2\binom{d}{2} + d$. Such expressions will likely lead you to different but equivalent sums.

Bonus:

If instead of summing MRs we will just count the number of scenarios then this will be well known problem about putting n indistinguishable balls into k distinguishable bins (a.k.a. "stars and bars"), where each ball is one dollar and i-th bin collects dollars that will be devoted to buying i-th magnifying glass. That tells us that it might be a good idea to put $n+k-1$ objects in a sequence and declare that n of them are balls and $k-1$ of them are walls between consecutive bins (in this interpretation we allow bins with 0 balls as well) and adjust that thinking to the original question.

What does $\binom{d+1}{2}$ stand for? Of course choosing 2 objects out of $d+1$. If this had been $\binom{d}{2}$ then we would have been able to say that, in addition to choosing walls, on every segment of balls between two consecutive walls, we choose two of them as special ones. Because of that we can view total sum of MRs over all scenarios as a number of scenarios where we partition sequence of $n+k-1$ objects into $k-1$ walls, $2k$ special balls and $n-2k$ ordinary balls. However we know that if we erase ordinary balls and leave walls and special balls only, then we will get sequence 2 special balls, 1 wall, 2 special balls, 1 wall, ..., 1 wall, 2 special balls. Because of that it suffices to choose $3k-1$ objects that are of type "either wall or special ball" and that uniquely determines positions of walls and special balls. Hence the result will be $\binom{n+k-1}{3k-1}$ in that case.

However we changed $\binom{d+1}{2}$ into $\binom{d}{2}$. What is the answer for the original version then? Well, if numbers of balls in bins are respectively $i_1, \ldots, i_k$ and $i_1 + \ldots + i_k = n$ then $(i_1+1) + \ldots + (i_k+1) = n+k$. Because of that there is a bijection between scenarios where total number of balls is n and where total number of balls is $n+k$, but we require each bin to have at least one ball. However if there is an empty bin then it adds nothing to the result as we cannot identify two special balls between consecutive walls. So we can apply previous reasoning for n increased by k and get that result is $\binom{(n+k)+k-1}{3k-1} = \binom{n+2k-1}{3k-1}$.