Introduction to Combinatorics
Generating functions

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1. (a) Consider the lowest level to which the partial sum drops down.

   (b) Observe that the nth Catalan number is the number of sequences $s_1, \ldots, s_{2n+1} \in \{-1, 1\}$ having all partial sums strictly positive and satisfying $\sum_{i=1}^{2n+1} s_i = 1$

2. Build a bijection between $S_n^{(132)}$ and Dyck paths based on position of n is permutation π.

3. (a) Use reflection principle.

   (b), (c) Follow easily from (a).

4. Create generating functions $R(x) = \sum_{i=0}^{\infty} R_i x^i$ and $S(x) = \sum_{i=0}^{\infty} S_i x^i$. Express both of them as products of various expressions and prove that they are in fact equal.

   Bonus: Even though combinatorial approach may be hard to spot from scratch, analyze what really happens in the proof with generating functions and try getting some combinatorial insight from it.

5. a) Try using general formula for exponentiating generating functions:

$$[x^n](c_0x^0 + c_1x^1 + \ldots)^k = \sum_{\substack{i_1,\ldots,i_k \geq 0 \\ i_1+\ldots+i_k=n}} c_{i_1}c_{i_2}\ldots c_{i_k}$$

($[x^n]F(x)$ stands for a coefficient next to x^n in $F(x)$) as well as well known formula: $\frac{1}{(1-x)^c} = \sum_{i=0}^{\infty} x^i \binom{i+c-1}{c-1}$

b) Try reducing this question to the previous one by expressing n^2 as a linear combination of Newton's symbols.

Bonus) Let's first think about counting plain number of scenarios. What does it resemble then? Moreover, how can you incorporate $\binom{d+1}{2}$ MRs of individual glasses into it? Maybe it will be a bit easier to start with $\binom{d}{2}$ instead of $\binom{d+1}{2}$ and then adjust the argument.