Introduction to Combinatorics
Spectral graph theory

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1. Let G be a d regular graph on n vertices. Let $\lambda_1 \geq \ldots \geq \lambda_n$ be eigenvalues of the adjacency matrix of G.
   (a) Show that $\lambda_n \geq -d$.
   (b) Show that $\lambda_n = -d$ if and only if at least one connected component of G is bipartite.
2. Let G be a directed graph on n vertices (possibly with loops), M be its adjacency matrix and let k be a non-negative integer. Prove that $(M^k)_{u,v}$ is the number of walks ("walk" is "marszruta" in Polish) of length k from u to v, where walk of length k from u to v is defined as a sequence of vertices $w_0, w_1, \ldots, w_k$, where $w_0 = i$, $w_k = j$ and there is an edge from w_i to w_{i+1} for all valid i (walks can be thought of as paths with not necessarily distinct vertices and edges). Conclude that $tr(M^k)$ is the number of closed walks of length k in G.
3. We define a sequence of matrices by taking $A_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $A_n = \begin{bmatrix} A_{n-1} & I \\ I & -A_{n-1} \end{bmatrix}$ for $n \geq 2$. Show that A_n has eigenvalues $\sqrt{n}$ and $-\sqrt{n}$ both with multiplicities 2^{n-1}.
4. Let $G = (V, E)$ be a simple graph and let $\Delta(G)$ be its maximal vertex degree. Suppose that a symmetric matrix A whose rows and columns are indexed by V has the following property: for every $u, v \in V$ we have $A_{uv} \in \{-1, 0, 1\}$ and moreover $A_{uv} = 0$ whenever u, v are non-adjacent. Show that for any eigenvalue λ of A we have $|\lambda| \leq \Delta(G)$.
5. Let $G_n = (V, E)$ be the hypercube graph, that is $V = \{0, 1\}^n$ and $x, y \in E$ if and only if $|x - y| = 1$. Let H be a subgraph of G_n induced by a set of cardinality $2^{n-1} + 1$. Show that the maximal vertex degree of H is at least $\sqrt{n}$.

   Comment: This was a long standing open problem that has been resolved a year ago, but with all the earlier preparation you should be ready to give it a try.
6. Clebsch graph is a unique graph G (up to isomorphism) on 16 vertices, such that it is 5-regular, for every pair of adjacent vertices they have no common neighbours and for every pair of nonadjacent vertices they have exactly 2 common neighbours. Determine the multiset of eigenvalues of its adjacency matrix.

   Comment: This is of course a finite problem, where you can take this matrix and use some computation tool to solve it for you, but of course that is not the point. Doing it in a not brute-forced way can teach you some tricks that may be useful in general.
7. Prove that the number of $n \times n$ binary matrices whose all eigenvalues are real and positive is equal to the number of directed acyclic graphs on n vertices.