Introduction to Combinatorics

Spectral graph theory

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1. We have

$$d - \lambda_n = \max_{x \neq 0} \frac{x^T(dI - A)x}{x^T x}$$

and

$$x^T(dI - A)x = \sum_{\{u,v\} \in E} (x_u - x_v)^2 = d|x|^2 - 2 \sum_{\{u,v\} \in E} x_u x_v = 2d|x|^2 - \sum_{\{u,v\} \in E} (x_u + x_v)^2.$$

Thus,

$$d - \lambda_n = \max_{x \neq 0} \left(2d - \frac{\sum_{\{u,v\} \in E} (x_u + x_v)^2}{x^T x} \right) \leq 2d.$$

We get $\lambda_n \geq -d$. Moreover, if $\lambda_n = -d$ then there must be a non-zero vector x such that

$$\sum_{\{u,v\} \in E} (x_u + x_v)^2 = 0.$$

Let v_0 be a vertex with $x_{v_0} = a \neq 0$. Define

$$A = \{v : \ x_v = a\}, \qquad B = \{v : \ x_v = -a\}, \qquad R = \{v : \ |x_v| \neq a\}.$$

We see that $A \cup B$ is disconnected from the rest of the graph R. Otherwise any edge $\{u, v\}$ from R to $A \cup B$ would give $(x_u + x_v)^2 > 0$. Moreover, for the same reason if $v \in A$ and $\{u, v\} \in E$ then $u \in B$. Thus, A and B gives a bipartition of $A \cup B$, which is a sum of connected bipartite components of G.

2. We will use induction. It is trivial for $k = 0$. Let us now assume that our thesis is true for $k-1$ and we will prove it for k. We know that $(M^k)_{u,v} = \sum_{w=1}^{n} (M^{k-1})_{u,w} M_{w,v}$. We can observe that $(M^{k-1})_{u,w} M_{w,v}$ is exactly the number of walks from u to v such that w is the second to last vertex on this walk and from that we conclude the thesis.

3. We show by induction that $A_n^2 = nI$. This is clearly true for $n = 1$. Assume $A_{n-1}^2 = (n-1)I$. Then

$$A_n^2 = \begin{bmatrix} A_{n-1} & I \\ I & -A_{n-1} \end{bmatrix} \cdot \begin{bmatrix} A_{n-1} & I \\ I & -A_{n-1} \end{bmatrix} = \begin{bmatrix} A_{n-1}^2 + I & 0 \\ 0 & A_{n-1}^2 + I \end{bmatrix} = \begin{bmatrix} nI & 0 \\ 0 & nI \end{bmatrix} = nI.$$

Let $\lambda_1, \ldots, \lambda_{2^n}$ be the eigenvalues of A_n. Then $\lambda_1^2, \ldots, \ldots \lambda_{2^n}^2$ are the eigenvalues of $A_n^2 = nI$ and thus $\lambda_i^2 = n$ for all $i = 1, \ldots, 2^n$. Thus $\lambda_i \in \{-\sqrt{n}, \sqrt{n}\}$ for all $i = 1, \ldots, 2^n$. The fact that we have precisely 2^{n-1} eigenvalues equal $\sqrt{n}$ follows from the fact that $\mathrm{tr}(A_n) = 0$.

4. *Solution 1.*

Suppose G has n vertices. Let v be the eigenvector corresponding to the eigenvalue λ. Without loss of generality we assume that $v \neq 0$. Without loss of generality we assume that v_1 (the first coordinate of v) is the largest in absolute value among all coordinates of v. We therefore get

$$|\lambda v_1| = |(Av)_1| = \left|\sum_{j=1}^{n} A_{1j} v_j\right| \leq \sum_{j \sim 1} |v_j| \leq \Delta(G)|v_1|.$$

Thus $|\lambda| \leq \Delta(G)$.

Solution 2.

We know that for any eigenvalue λ of A we have $\lambda \leq \max_{x \neq 0} \frac{\langle Ax, x\rangle}{\langle x, x\rangle}$. Hence, if we show that $\frac{\langle Ax, x\rangle}{\langle x, x\rangle} \leq \Delta(G)$ then we will show that $\lambda \leq \Delta(G)$ (proof that $-\lambda \leq \Delta(G)$ will be very similar and the thesis will follow). However,

$$\frac{\langle Ax, x\rangle}{\langle x, x\rangle} \leq \Delta(G) \Leftrightarrow \langle Ax, x\rangle \leq \Delta(G)\langle x, x\rangle \Leftrightarrow \langle Ax, x\rangle \leq \langle (\Delta(G)I)x, x\rangle \Leftrightarrow 0 \leq \langle (\Delta(G)I - A)x, x\rangle$$

But if we will expand $\langle (\Delta(G)I - A)x, x\rangle$ then we will get

$$\sum_{i=1}^{n} \Delta(G) x_i^2 - 2 \sum_{1 \leq u < v \leq n} A_{uv} x_u x_v.$$

Second sum will not change if we sum over only such pairs (u, v) that $u < v$ and $uv \in E$. We can now express this as a sum of squares. Let us define $p(x, y, s)$ as $x^2 + y^2 - 2sxy$. $p(x, y, s)$ is nonnegative for $s \in \{-1, 0, 1\}$ as $p(x, y, 1) = (x - y)^2, p(x, y, 0) = x^2 + y^2, p(x, y, -1) = (x + y)^2$.

We can now express above sum as

$$\sum_{i=1}^{n} (\Delta(G) - deg(v_i)) x_i^2 + \sum_{u < v, uv \in E} p(x_u, x_v, A_{uv}),$$

where $deg(v_i)$ denotes degree of i-th vertex. Since $\Delta(G)$ was maximum degree, we know that $\Delta(G) - deg(v_i) \geq 0$ and we can conclude that this sum is nonnegative. As mentioned before, we can create a very similar proof that $-\lambda \leq \Delta(G)$ and the thesis follows.

5. Let A_n be the matrix from Exercise 3. We claim that if in A_n we replace all the entries -1 with 1 we get the adjacency matrix B_n of G. To see this just divide the cube into two subcubes $\{x_1 = 0\}$ and $\{x_1 = 1\}$ and observe that the adjacency matrix B_n of G_n has upper left and lower right $2^{n-1} \times 2^{n-1}$ submatrices corresponding to the left and right subcube an therefore equal to B_{n-1} and the upper right and lower left $2^{n-1} \times 2^{n-1}$ submatrices corresponding to the matching between both subcubes and therefore equal to I. Thus $B_n = \begin{bmatrix} B_{n-1} & I \\ I & B_{n-1} \end{bmatrix}$ and the claim follows.

   Thus the matrix A_n satisfies assumptions of Exercise 4. Let H be a subgraph induced by $2^{n-1} + 1$ vertices. Let A_H be the corresponding submatrix of A_n. The matrix A_H also satisfies assumptions of Exercise 4. Thus $\Delta(H) \geq \lambda_1(A_H)$. By the Cauchy interlacing principle $\lambda_1(A_H) \geq \lambda_{2^n-(2^{n-1}+1)+1}(A_n) = \lambda_{2^{n-1}}(A_n) = \sqrt{n}$, where the last equality follows from Exercise 3.

6. Let M be its adjacency matrix. We can see that mentioned conditions mean exactly the fact that $M^2 + 2M = 3I + 2J$ (where J is a matrix consisting of all ones). We know that if $\lambda_1, \ldots, \lambda_n$ is a multiset of eigenvalues of M then $\lambda_1^2 + 2\lambda_1, \ldots, \lambda_n^2 + 2\lambda_n$ is a multiset of eigenvalues of $M^2 + 2M$, so getting to know eigenvalues of $M^2 + 2M$ may be helpful.

   We can now focus on determining characteristic polynomial of $M^2 + 2M$ as multiset of its roots is multiset of its eigenvalues. $det(M^2 + 2M - xI) = det(2J + (3 - x)I)$. If we now subtract last row of $2J + (3 - x)I$ from all its other rows, then we will get a very sparse matrix whose determinant will be easy to compute. This matrix A fulfills:

$$A_{ij} = \begin{cases} 3 - x, & \text{if } i = j < n \\ 5 - x, & \text{if } i = j = n \\ 2, & \text{if } i = n, j < n \\ x - 3, & \text{if } j = n, i < n \\ 0, & \text{otherwise} \end{cases}$$

   where $n = 16$. Using permutation formula for determinant it is easy to compute that its determinant is $(5 - x)(3 - x)^{n-1} - (n - 1) \cdot 2 \cdot (x - 3) \cdot (3 - x)^{n-2} = (3 - x)^{15}(35 - x)$

   Hence, eigenvalues of matrix A are one times 35 and fifteen times 3 and the same goes for $M^2 + 2M$. Because of that, one eigenvalue λ of M fulfills $\lambda^2 + 2\lambda = 35$ and fifteen of them fulfill $\lambda^2 + 2\lambda = 3$. We know that since Clebsch graph is 5-regular then 5 is one of its eigenvalues and that is the one fulfilling $\lambda^2 + 2\lambda = 35$. All other eigenvalues come from the set $\{-3, 1\}$ as there are the only real values that satisfy $\lambda^2 + 2\lambda = 3$. Because of that it suffices to determine their multiplicities. However $0 = tr(M) = \lambda_1 + \lambda_2 + \ldots + \lambda_{16} = 5 + c_1 - 3c_{-3}$, where c_1 and c_{-3} denote multiplicities of 1 and -3. Moreover $c_1 + c_{-3} = 15$. We deduce that $c_1 = 10$ and $c_{-3} = 5$.

7. Let M be a binary matrix with all its eigenvalues real and positive and let $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n > 0$ be these eigenvalues. We know that $0 \leq \sum_{i=1}^n \lambda_i \leq n$ since $\sum_{i=1}^n \lambda_i = tr(M)$ and that $\prod_{i=1}^n \lambda_i = det(M)$. We know that $\prod_{i=1}^n \lambda_i$ is positive and that $det(M)$ is integer, so $\prod_{i=1}^n \lambda_i \geq 1$. However, based on inequality between arithmetic and geometric mean we can conclude that $1 \geq \frac{\sum_{i=1}^n \lambda_i}{n} \geq \sqrt[n]{\prod_{i=1}^n \lambda_i} \geq 1$. Because of that we conclude that $\lambda_1 = \ldots = \lambda_n = 1$. In particular $tr(M) = n$, so $m_{1,1} = \ldots = m_{n,n} = 1$.

   $\lambda_1^k, \ldots, \lambda_n^k$ are eigenvalues of M^k, so we can deduce that $n = \sum_{i=1}^n \lambda_i^k = tr(M^k)$. Since $m_{i,i} = 1$ we know that all diagonal entries of M^k are at least 1, but their sum is equal to n, so in fact all of them need to be exactly 1.

   Let G be a directed graph such that for all (i,j) we put an edge from i to j if and only if $M_{i,j} = 1$ (in particular this graph contains n loops). Based on exercise 2 $(M^k)_{i,j}$ is equal to the number of walks of length k from i to j. In particular $(M^k)_{i,i}$ is equal to the number of closed walks (i.e. not necessarily simple cycles) from i to i of length k. One such walk consists of using k times loop in i-th vertex and we know that $w_{i,i} = 1$, so there are no other closed walks from i to i of length k. Hence, if we remove loops from G then G becomes acyclic.

   We shall now prove that all matrices corresponding to acyclic directed graphs (with ones on diagonal) have n eigenvalues equal to 1. Directed acyclic graphs admit topological order, i.e. there exists a permutation of their vertices, so that edges go only from vertices that are earlier in this order to vertices that are later. That means that if we apply that permutation to both rows and columns, we will get upper-triangular matrix with ones on diagonal (and applying such permutation doesn't change the multiset of eigenvalues). Characteristic polynomial of such matrix is $(1-x)^n$ what means that such matrix has n eigenvalues which are equal to one.

   We obtained a bijection between binary $n \times n$ matrices with all their eigenvalues which are real and positive and directed acyclic graphs on n vertices, hence their numbers are equal.