Introduction to Combinatorics

Spectral graph theory – hints

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1. (a) the proof is similar to that of Lemma 7 in the lecture notes.

   (b) Consider the expression $\sum_{\{u,v\}\in E}(x_u+x_v)^2$. What does the Courant-Fisher-Weyl principle for the largest eigenvalue of $dI - A$ tell you?

2. Use induction. Split the path by guessing second to last vertex.

3. Show that $A_n^2 = nI$

4. Consider the eigenvector associated with λ and take its entry with the largest absolute value. Or view this problem as a generalization of facts that $\lambda_1 \leq d$ and $\lambda_n \geq -d$ in d-regular graphs and try to adjust their proofs.

5. Use Exercise 3, Exercise 4 and Cauchy interlacing theorem.

6. If M is a matrix in question, what can you say about $M^2 + 2M$? How its eigenvalues relate to the eigenvalues of M?

7. You should use known facts about eigenvalues. If M is a matrix and $\lambda_1, \ldots, \lambda_n$ are its eigenvalues (with corresponding multiplicities) then

   - $\lambda_1 + \ldots + \lambda_n = tr(M)$
   - $\lambda_1 \lambda_2 \ldots \lambda_n = det(M)$
   - $\lambda_1^k, \ldots, \lambda_n^k$ are eigenvalues of M^k

   Exercise 2 may be helpful as well.