The unconditional case of the complex S-inequality

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Abstract

In this note we prove the complex counterpart of the S-inequality for complete Reinhardt sets. In particular, this result implies that the complex S-inequality holds for unconditional convex sets. As a by-product we also obtain the S-inequality for the exponential measure in the unconditional case.

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1 Introduction

Studying various aspects of a Gaussian measure in a Banach space one often needs precise estimates on measures of balls and their dilations. This gives raise to the question how the function $(0, \infty) \ni t \mapsto \mu(tB)$ behaves. Here B is a convex and symmetric subset of some Banach space, i.e. an unit ball with respect to some norm, and μ is a Gaussian measure. Thanks to certain approximation arguments we may only deal with the simplest spaces, namely $\mathbb{R}^n$ or $\mathbb{C}^n$. In the former case the issue is well understood due to R. Latała and K. Oleszkiewicz. Denote by γ_n the standard Gaussian measure on $\mathbb{R}^n$, i.e. the measure with the density at a point $(x_1, \ldots, x_n)$ equal to $\frac{1}{\sqrt{2\pi}^n} \exp\left(-x_1^2/2 - \ldots - x_n^2/2\right)$. In [LO1] it is shown that for a symmetric convex body $K \subset \mathbb{R}^n$ and the strip $P = \{x \in \mathbb{R}^n \mid |x_1| \leq p\}$, where p is chosen so that $\gamma_n(K) = \gamma_n(P)$, we have

$$\gamma_n(tK) \geq \gamma_n(tP), \qquad t \geq 1.$$

This result is called *S-inequality.* The interested reader is also referred to the concise survey [Lat].

In the present note we would like to focus on S-inequality for sets which correspond to unit balls with respect to unconditional norms on $\mathbb{C}^n$. Some partial results concerning the general case has been recently obtained in [Tko].

Definitions and preliminary statements are provided in Section 2. Section 3 is devoted to the main result. It also contains a proof of a one-dimensional inequality, which bounds entropy, and seems to be the heart of the proof of our main theorem.

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2 Preliminaries

We define the standard Gaussian measure ν_n on the space $\mathbb{C}^n$ via the formula

$$\nu_n(A) = \gamma_{2n}\left(\tau(A)\right), \qquad \text{for any Borel set } A \subset \mathbb{C}^n,$$

where $\mathbb{C}^n \overset{\tau}{\longmapsto} \mathbb{R}^{2n}$ is the bijection given by

$$\tau(z_1, \ldots, z_n) = (\mathfrak{Re} z_1, \mathfrak{Im} z_1, \ldots, \mathfrak{Re} z_n, \mathfrak{Im} z_n).$$

We adopt the notation $\mathbb{R}_+ = [0, +\infty)$. Later on we will also extensively use the notion of the *entropy* of a function $f\colon X \longrightarrow \mathbb{R}_+$ with respect to a probability measure μ on a measurable space X

$$\operatorname{Ent}_\mu f = \int_X f(x) \ln f(x) \mathrm{d}\mu(x) - \left(\int_X f(x) \mathrm{d}\mu(x)\right) \ln \left(\int_X f(x) \mathrm{d}\mu(x)\right). \quad (1)$$

We say that a closed subset K of $\mathbb{C}^n$ *supports the complex S-inequality*, *$S\mathbb{C}$-inequality* for short, if any its dilation $L = sK$, $s > 0$, and any *cylinder* $C = \{z \in \mathbb{C}^n \mid |z_1| \le R\}$ satisfy

$$\nu_n(L) = \nu_n(C) \quad \Longrightarrow \quad \nu_n(tL) \ge \nu_n(tC), \quad \text{for } t \ge 1. \quad (2)$$

Note that the natural counterpart of S-inequality in the complex case is the following conjecture due to Prof. A. Pełczyński, which has already been discussed in [Tko].

Conjecture. *All closed subsets K of $\mathbb{C}^n$ which are* rotationally symmetric*, that is $e^{i\theta}K = K$ for any $\theta \in \mathbb{R}$, support $S\mathbb{C}$-inequality.*

In the present paper we are interested in the class $\mathfrak{R}$ of all closed sets in $\mathbb{C}^n$ which are *Reinhardt complete*, i.e. along with each point $(z_1, \ldots, z_n)$ such a set contains all points $(w_1, \ldots, w_n)$ for which $|w_k| \le |z_k|$, $k = 1, \ldots, n$ (consult for instance the textbook [Sh, I.1.2, pp. 8–9]). The key point is that this class contains all unit balls with respect to unconditional norms on $\mathbb{C}^n$. Recall that a norm $\|\cdot\|$ is said to be *unconditional* if $\|(e^{i\theta_1}z_1, \ldots, e^{i\theta_n}z_n)\| = \|z\|$ for all $z \in \mathbb{C}^n$ and $\theta_1, \ldots, \theta_n \in \mathbb{R}$.

The goal is to prove that all sets from the class $\mathfrak{R}$ support $S\mathbb{C}$-inequality. Now we establish some general yet simple observations which allow us to reduce the problem to a one-dimensional entropy inequality.

Proposition 1. *A closed subset K of $\mathbb{C}^n$ supports $S\mathbb{C}$-inequality if and only if for any its dilation L and any cylinder C we have*

$$\nu_n(L) = \nu_n(C) \quad \Longrightarrow \quad \frac{\mathrm{d}}{\mathrm{d}t}\nu_n(tL)\Big|_{t=1} \ge \frac{\mathrm{d}}{\mathrm{d}t}\nu_n(tC)\Big|_{t=1}. \quad (3)$$

Proof. We are only to show the interesting part that (3) implies (2) following the proof of [KS, Lemma 1]. Fix a dilation L of K and a cylinder C such that $\nu_n(L) = \nu_n(C)$. Let a function h be given by $\nu_n(tL) = \nu_n(h(t)C)$, $t \ge 1$. Then, by the assumption, we find

$$h(t)\frac{\mathrm{d}}{\mathrm{d}s}\nu_n(sC)\Big|_{s=h(t)} = \frac{\mathrm{d}}{\mathrm{d}s}\nu_n(sh(t)C)\Big|_{s=1} \le \frac{\mathrm{d}}{\mathrm{d}s}\nu_n(stL)\Big|_{s=1} = t\frac{\mathrm{d}}{\mathrm{d}s}\nu_n(sL)\Big|_{s=t}.$$

Yet, differentiating the equation which defines the function h we get $\frac{\mathrm{d}}{\mathrm{d}s}\nu_n(sL)\big|_{s=t} = h'(t)\frac{\mathrm{d}}{\mathrm{d}s}\nu_n(sC)\big|_{s=h(t)}$, thus $h(t) \le th'(t)$. It means that the function $h(t)/t$ is nondecreasing, so $1 = h(1) \le h(t)/t$ for $t \ge 1$. $\square$

For any closed set A the derivative of the function $t \mapsto \nu_n(tA)$ is easy to compute. Indeed,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\nu_n(tA)\bigg|_{t=1} &= \frac{\mathrm{d}}{\mathrm{d}t}\int_{tA} e^{-|z|^2/2}\mathrm{d}z\bigg|_{t=1} = \frac{\mathrm{d}}{\mathrm{d}t}\int_{A} t^{2n}e^{-t^2|w|^2/2}\mathrm{d}w\bigg|_{t=1}\\ &= 2n\nu_n(A) - \int_A |z|^2\mathrm{d}\nu_n(z).\end{aligned}$$

Moreover, the integral of $|z|^2$ over a cylinder C may be expressed explicitly in terms of the measure $\nu_n(C)$. Namely,

$$\int_C |z|^2\mathrm{d}\nu_n(z) = 2(1-\nu_n(C))\ln\left(1-\nu_n(C)\right) + 2n\nu_n(C).$$

Combining these two remarks with the preceding proposition we obtain an equivalent formulation of the problem.

Proposition 2. *A closed subset K of $\mathbb{C}^n$ supports $S\mathbb{C}$-inequality if and only if for any its dilation L*

$$\int_L |z|^2\mathrm{d}\nu_n(z) \le 2n\nu_n(L) + 2(1-\nu_n(L))\ln\left(1-\nu_n(L)\right). \tag{4}$$

3 Main result

We aim at proving the aforementioned main result, which reads as follows

Theorem 1. *Any set from the class $\mathfrak{R}$ supports $S\mathbb{C}$-inequality.*

We begin with a one-dimensional entropy inequality.

Lemma 1. *Let μ be a Borel probability measure on $\mathbb{R}_+$ and suppose $f\colon \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is a bounded and non-decreasing function. Then*

$$\mathrm{Ent}_\mu f \le -\int_{\mathbb{R}_+} f(x)\Big(1 + \ln\mu\left((x,\infty)\right)\Big)\mathrm{d}\mu(x). \tag{5}$$

Proof. Using homogeneity of both sides of (5), without loss of generality, we can assume that $\int_{\mathbb{R}_+} f\mathrm{d}\mu = 1$. Then we may rewrite the assertion of the lemma as follows

$$\int_{\mathbb{R}_+} \ln\left(f(x)\int_{(x,\infty)}\mathrm{d}\mu(t)\right)f(x)\mathrm{d}\mu(x) \le -1.$$

Introduce the probability measure ν on $\mathbb{R}_+$ with the density f with respect to μ. Thanks to the monotonicity of f we can bound the left hand side of the last inequality by

$$\int_{\mathbb{R}_+} \ln\Big(\nu\left((x,\infty)\right)\Big)\mathrm{d}\nu(x) = -\int_0^\infty\int_0^1 \frac{\mathrm{d}u}{u}\mathbf{1}_{\{u \ge \nu((x,\infty))\}}(u,x)\mathrm{d}\nu(x).$$

Define the function

$$H(y) := \inf \{t \mid \nu((t,\infty)) \le y\},$$

which is the *inverse* tail function, and observe that

$$\{(u,x) \mid u \ge \nu((x,\infty))\} \supset \{(u,x) \mid H(u) \le x\},$$

as $u \ge \nu((H(u),\infty)) \ge \nu((x,\infty))$. This leads to

$$\begin{aligned}-\int_0^\infty \int_0^1 \frac{\mathrm{d}u}{u} \mathbf{1}_{\{u \ge \nu((x,\infty))\}}(u,x)\mathrm{d}\nu(x) &\le -\int_0^\infty \int_0^1 \frac{\mathrm{d}u}{u} \mathbf{1}_{\{H(u)\le x\}}(u,x)\mathrm{d}\nu(x) \\ &= -\int_0^1 \nu([H(u),\infty)) \frac{\mathrm{d}u}{u}.\end{aligned}$$

Since $u \le \nu([H(u),\infty))$, we finally get the desired estimation. □

Now, for a certain class of functions, we establish the multidimensional version of inequality (5). For the simplicity, we formulate this result for the Gaussian measure.

Lemma 2. *Let $g\colon \mathbb{C}^n \longrightarrow \mathbb{R}_+$ be a bounded function satisfying*

1) $g((e^{i\theta_1}z_1,\dots,e^{i\theta_n}z_n)) = g(z)$ for any $z \in \mathbb{C}^n$ and $\theta_1,\dots,\theta_n \in \mathbb{R}$,

2) for any $w, z \in \mathbb{C}^n$ the condition $|w_k| \le |z_k|$, $k = 1,\dots,n$ implies $g(w) \le g(z)$.

Then

$$\mathrm{Ent}_{\nu_n} g \le \int_{\mathbb{C}^n} g(z)\left(\frac{|z|^2}{2} - n\right)\mathrm{d}\nu_n(z). \tag{6}$$

Proof. One piece of notation: for a fixed vector $r = (r_1,\dots,r_n) \in (\mathbb{R}_+)^n$ we denote $r^k = (r_1,\dots,r_{k-1},r_{k+1},\dots,r_n) \in (\mathbb{R}_+)^{n-1}$, and then define the functions

$$g_k^{r^k}(x) = g(r_1,\dots,r_{k-1},x,r_{k+1},\dots,r_n), \qquad k = 1,\dots,n.$$

Notice that for a function $h\colon \mathbb{C} \longrightarrow \mathbb{R}_+$ obeying the property 1) we get

$$\int_{\mathbb{C}} h(z)\mathrm{d}\nu_1(z) = \frac{1}{2\pi}\int_0^{2\pi}\int_0^\infty h(re^{i\theta})e^{-r^2/2}r\mathrm{d}r\mathrm{d}\theta = \int_0^\infty h(r)\mathrm{d}\mu(r),$$

where μ denotes the probability measure on $\mathbb{R}_+$ with the density at r given by $re^{-r^2/2}$. Therefore

$$\begin{aligned}\int_{\mathbb{C}^n} g(z)\left(\frac{|z|^2}{2} - n\right)\mathrm{d}\nu_n(z) &= \int_{(\mathbb{R}_+)^n} g(r)\left(\frac{\sum_{k=1}^n r_k^2}{2} - n\right)\mathrm{d}\mu^{\otimes n}(r) \\ &= \int_{(\mathbb{R}_+)^n}\sum_{k=1}^n\left[\int_{\mathbb{R}_+} g_j^{r^j}(x)\left(\frac{x^2}{2} - 1\right)\mathrm{d}\mu(x)\right]\mathrm{d}\mu^{\otimes n}(r).\end{aligned}$$

Applying Lemma 1 for the function $g_j^{r^j}$ and the measure μ we obtain the estimation

$$\begin{aligned}\int_{\mathbb{C}^n} g(z)\left(\frac{|z|^2}{2} - n\right)\mathrm{d}\nu_n(z) &\ge \int_{(\mathbb{R}_+)^n}\sum_{k=1}^n \mathrm{Ent}_\mu\, g_j^{r^j}\,\mathrm{d}\mu^{\otimes n}(r) \\ &\ge \mathrm{Ent}_{\mu^{\otimes n}}\, g = \mathrm{Ent}_{\nu_n}\, g,\end{aligned}$$

where the last inequality follows from subadditivity of entropy (for example see [Led, Proposition 5.6]). □

Proof of Theorem 1. Fix $K \in \mathfrak{R}$. In order to show (4) we introduce the function $g(z) = 1 - \mathbf{1}_K(z)$. We adopt the standard convention that $0 \ln 0 = 0$, hence the desired inequality is equivalent to (6). Thus the application of Lemma 2 for the function g finishes the proof. $\square$

Theorem 1 immediately implies that the Cartesian products of cylinders support $S\mathbb{C}$-inequality. As a consequence, $S\mathbb{C}$-inequality possesses a tensorization property.

Corollary 1. *Assume sets $K_1 \subset \mathbb{C}^{n_1}, \ldots, K_\ell \subset \mathbb{C}^{n_\ell}$ support $S\mathbb{C}$-inequality. Then the set $K_1 \times \ldots \times K_\ell$ also supports $S\mathbb{C}$-inequality.*

Another consequence of the main theorem concerns the standard exponential measure λ_n on $\mathbb{R}^n$, i.e.

$$\mathrm{d}\lambda_n(x) = \frac{1}{2^n} e^{-|x|_1} \mathrm{d}x, \qquad x \in \mathbb{R}^n,$$

where we denote $|(x_1, \ldots, x_n)|_1 = \sum_{i=1}^n |x_i|$. It turns out that certain subsets of $\mathbb{R}^n$ support the S-inequality for λ_n with *strips* as the optimal sets. To state the result a few definitions will be useful. We say that a set $K \subset (\mathbb{R}_+)^n$ is an *ideal* if along with any its point $x \in K$ it contains the cube $[0, x_1] \times \ldots \times [0, x_n]$. A set $K \subset \mathbb{R}^n$ is called *unconditional* if $(\epsilon_1 x_1, \ldots, \epsilon_n x_n) \in K$ whenever $(x_1, \ldots, x_n) \in K$ and $\epsilon_1, \ldots, \epsilon_n \in \{-1, 1\}$. By an *unconditional ideal* K in $\mathbb{R}^n$ we mean the unconditional set K such that the set $K \cap (\mathbb{R}_+)^n$ is an ideal. For instance, any unconditional convex set is also an unconditional ideal.

Theorem 2. *For any closed unconditional ideal $K \subset \mathbb{R}^n$ and for any strip $P = \{x \in \mathbb{R}^n \mid |x_1| \le p\}$, $p \ge 0$, we have*

$$\lambda_n(K) = \lambda_n(P) \quad \Longrightarrow \quad \forall t \ge 1 \ \lambda_n(tK) \ge \lambda_n(tP), \tag{7}$$

and, equivalently,

$$\lambda_n(K) = \lambda_n(P) \quad \Longrightarrow \quad \forall t \le 1 \ \lambda_n(tK) \le \lambda_n(tP). \tag{8}$$

Proof. The equivalence between (7) and (8) is straightforward. For instance, assume the latter does not hold. Then, there is $t_0 < 1$ such that $\lambda_n(t_0 K) > \lambda_n(t_0 P)$. We can find $s_0 < 1$ for which $\lambda_n(s_0 t_0 K) = \lambda_n(t_0 P)$. Using (7) we get a contradiction

$$\lambda_n(K) > \lambda_n(s_0 K) = \lambda_n\left(\frac{1}{t_0}(s_0 t_0 K)\right) \ge \lambda_n\left(\frac{1}{t_0}(t_0 P)\right) = \lambda_n(P) = \lambda_n(K).$$

Consider the mapping $F \colon \mathbb{C}^n \longrightarrow (\mathbb{R}_+)^n$ given by the formula

$$F(z_1, \ldots, z_n) = (|z_1|, \ldots, |z_n|).$$

Observe that for an ideal $A \subset (\mathbb{R}_+)^n$, the set $F^{-1}(A)$ is Reinhardt complete and integrating using the polar coordinates we find that

$$\nu_n\left(F^{-1}(A)\right) = \int_A \prod_{i=1}^n r_i e^{-r_i^2/2} \mathrm{d}r_1 \ldots \mathrm{d}r_n.$$

Now, let us change the variables according to the mapping $G\colon (\mathbb{R}_+)^n \longrightarrow (\mathbb{R}_+)^n$,

$$G(x_1,\ldots,x_n) = \frac{1}{2}(x_1^2,\ldots,x_n^2).$$

We obtain

$$\nu_n\left(F^{-1}(A)\right) = \int_{G(A)} e^{-\sum_{i=1}^n x_i} \mathrm{d}x.$$

Since $G(A)$ is an ideal iff so is A, we infer that for any unconditional ideal $K \subset \mathbb{R}^n$

$$\lambda_n(K) = \nu_n(\widetilde{K}), \qquad \text{where} \qquad \widetilde{K} := G^{-1}F^{-1}\left(K \cap (\mathbb{R}_+)^n\right).$$

Moreover, for a strip $P = \{x \in \mathbb{R}^n \mid |x_1| \le p\}$, the set $\widetilde{P} \subset \mathbb{C}^n$ is a cylinder. Note also that $\widetilde{tK} = \sqrt{t}\widetilde{K}$. These observations combined with Theorem 1 yield the assertion. $\square$

Following the method of [LO1, Corollary 3] we obtain the result concerning the comparison of moments.

Corollary 2. *Let $\|\cdot\|$ be a norm on $\mathbb{R}^n$ which is unconditional, i.e.*

$$\|(\epsilon_1 x_1,\ldots,\epsilon_n x_n)\| = \|(x_1,\ldots,x_n)\|,$$

for any $x_j \in \mathbb{R}$ and $\epsilon_j \in \{-1,1\}$. Then for $p \ge q > 0$

$$\left(\int_{\mathbb{R}^n} \|x\|^p \mathrm{d}\lambda_n(x)\right)^{1/p} \le C_{p,q}\left(\int_{\mathbb{R}^n} \|x\|^q \mathrm{d}\lambda_n(x)\right)^{1/q}, \tag{9}$$

where the constant

$$C_{p,q} = \frac{\left(\int_{\mathbb{R}} |x|^p \mathrm{d}\lambda_1(x)\right)^{1/p}}{\left(\int_{\mathbb{R}} |x|^q \mathrm{d}\lambda_1(x)\right)^{1/q}} = \frac{(\Gamma(p+1))^{1/p}}{(\Gamma(q+1))^{1/q}}$$

is the best possible.

Proof. The proof hinges on the fact that a ball $K = \{x \in \mathbb{R}^n \ \|x\| \le t\}$ with respect to the norm $\|\cdot\|$ is a closed convex unconditional set, so that Theorem 2 can be applied. $\square$

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