A note on the rational cuspidal curves

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Abstract

In this short note we give an elementary combinatorial argument, showing that the conjecture of J. Fernández de Bobadilla, I. Luengo, A. Melle-Hernández, A. Némethi (see [BL], Conjecture 1.2) follows from Theorem 5.4 in [BL] in the case of rational cuspidal curves with two critical points.

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1 Introduction

In this short note we deal with irreducible algebraic curves $C \subset \mathbb{C}P^2$. Such a curve has a finite set of singular points $\{z_i\}_{i=1}^n$ such that a neighbourhood of each singular point intersects C in a cone on a link $K_i \subset S^3$. We would like to know what possible configurations of links $\{K_i\}_{i=1}^n$ arise in this way. We consider only the case in which each K_i is connected (in this case K_i is a knot), and thus C is a rational curve, meaning that there is a rational surjective map $\mathbb{C}P^1 \to C$. Such a curve is called rational cuspidal. We refer to [M] for a survey on rational cuspidal curves.

Suppose that z is a cuspidal singular point of a curve C and B is a sufficiently small ball around z. Let $\Psi(t) = (x(t), y(t))$ be a local parametrization of $C \cap B$ near z. For any polynomial $P(x, y)$ we look at the order at 0 of the analytic map $t \mapsto P(x(t), y(t)) \in \mathbb{C}$. Let S be the set of integers, which can be realized as the order for some P. Then S is a semigroup of $\mathbb{Z}_{\geq 0}$. We call it the semigroup of the singular point, see [W] for the details and proofs. The gap sequence, $G = \mathbb{Z}_{\geq 0} \backslash S$, has precisely $\mu/2$ elements, where the largest one is $\mu - 1$. Here μ stands for the Milnor number. Assume that K is the link of the singular point z. The Alexander polynomial of K can be written in the form

$$\Delta_K(t) = \sum_{i=0}^{2m} (-1)^i t^{n_i},$$

where $(n_i)_{i=0}^{2m}$ form an increasing sequence with $n_0 = 0$ and $n_{2m} = 2g$, for $g = g(K)$ being the genus of K. Writing $t^{2n_i} - t^{2n_{i-1}} = (t-1)(t^{2n_i - 1} + t^{2n_i - 2} + \ldots + t^{2n_{i-1}})$ yields the representation

$$\Delta_K(t) = 1 + (t-1)\sum_{j=1}^{k} t^{g_j}, \tag{1}$$

for some finite sequence $0 < g_1 < g_2 < \ldots < g_k$. We have the following lemma (see [W], Exercise 5.7.7), which relates the Alexander polynomial to the gap sequence of a singular point.

Lemma 1. *The sequence $g_1, \ldots, g_k$ in (1) is the gap sequence of the semi-group of the singular point. In particular, $k = |G| = \mu/2$, where μ is the Milnor number, so $|G|$ is the genus.*

If we write $t^{g_j} = (t-1)(t^{g_j - 1} + t^{g_j - 2} + \ldots + 1) + 1$, we obtain

$$\Delta_K(t) = 1 + (t-1)g(K) + (t-1)^2 \sum_{j=0}^{\mu-2} k_j t^j,$$

where $k_j = |\{m > j : m \notin S\}|$. This motivates the following definition.

Definition. For any finite increasing sequence of positive integers G we define

$$I_G(m) = |\{k \in G \cup \mathbb{Z}_{<0} : \ k \geq m\}|,$$

where $\mathbb{Z}_{<0}$ is the set of negative integers. We shall call I_G the gap function, because in most applications G will be a gap sequence of some semigroup.

Clearly, for $j = 0, 1, \ldots, \mu - 2$ we have $I_G(j+1) = k_j$.

In [FLMN] the following conjecture was proposed.

Conjecture 1. *Suppose that the rational cuspidal curve C of degree d has critical points $z_1, \ldots, z_n$. Let $K_1, \ldots, K_n$ be the corresponding links of singular points and let $\Delta_1, \ldots, \Delta_n$ be their Alexander polynomials. Let $g = \sum g(K_i)$. Let $\Delta = \Delta_1 \cdot \ldots \cdot \Delta_n$, expanded as*

$$\Delta(t) = 1 + \frac{(d-1)(d-2)}{2}(t-1) + (t-1)^2 \sum_{j=0}^{2g-2} k_j t^j$$

Then for any $j = 0, \ldots, d-3$ *we have* $k_{d(d-j-3)} \leq (j+1)(j+2)/2$*, with equality for* $n = 1$*.*

This conjecture was verified in the case $n = 1$ by Borodzik and Livingston, see [BL].

We define the infimum convolution of n functions.

Definition. Let $I_1, I_2, \ldots, I_n : \mathbb{Z} \to \mathbb{Z}_{\geq 0}$. We define

$$(I_1 \diamond I_2 \ldots \diamond I_n)(k) = \min_{\substack{k_1, k_2, \ldots, k_n \in \mathbb{Z} \\ k_1 + k_2 + \ldots + k_n = k}} \left(I_1(k_1) + I_2(k_2) + \ldots + I_n(k_n)\right).$$

In [BL] the authors proved the following theorem.

Theorem 1. *(see [BL, Theorem 5.4]) Let* C *be a rational cuspidal curve of degree* d*. Let* $I_1, \ldots, I_n$ *be the gap functions associated to each singular point on* C*. Then for any* $j \in \{-1, 0, \ldots, d-2\}$ *we have*

$$I_1 \diamond I_2 \diamond \ldots \diamond I_n(jd+1) = \frac{1}{2}(j-d+1)(j-d+2).$$

Note that we have $|G_1| + |G_2| + \ldots + |G_n| = \frac{(d-1)(d-2)}{2}$. Therefore, one can give an equivalent reformulation of the Conjecture 1.

Conjecture 2. *Suppose that the rational cuspidal curve* C *of degree* d *has critical points* $z_1, \ldots, z_n$*. Let* $K_1, \ldots, K_n$ *be the corresponding links of singular points and let* $\Delta_1, \ldots, \Delta_n$ *be their Alexander polynomials. Moreover, let* $G_1, G_2, \ldots, G_n$ *be the gap sequences of these points. Let* $g = |G_1| + |G_2| + \ldots + |G_n|$ *be the genus of* K*. Let* $\Delta = \Delta_1 \cdot \ldots \cdot \Delta_n$*, expanded as*

$$\Delta(t) = 1 + (t-1)g + (t-1)^2 \sum_{j=0}^{2g-2} k_j t^j$$

and let $I = I_1 \diamond I_2 \diamond \ldots \diamond I_n$*. Then for any* $j = 0, \ldots, d-3$ *we have* $k_{d(d-j-3)} \leq I(d(d-j-3)+1)$*, with equality for* $n = 1$*.*

In this note we give an elementary argument, showing that FLMN conjecture follows from Theorem 1 for $n = 2$. The idea of our proof is to forget about the specific structure of the problem coming from theory of singularities and to prove Conjecture 2 for general sets G_1, G_2. Namely, we have the following theorem.

Theorem 2. *Let G, H be two finite sets of positive integers and let $I_G, I_H : \mathbb{Z} \to \mathbb{Z}_{\geq 0}$ be their gap functions. Let us define the polynomials*

$$\begin{array}{l}\Delta_G(t) = 1 + (t-1)|G| + (t-1)^2 \sum_{j \geq 0} k_j^G t^j \\ \Delta_H(t) = 1 + (t-1)|H| + (t-1)^2 \sum_{j \geq 0} k_j^H t^j\end{array},$$

where $k_j^G = I_G(j+1), k_j^H = I_H(j+1)$, $j \geq 0$. Take $\Delta = \Delta_G \cdot \Delta_H$ and $I = I_G \diamond I_H$. Then

$$\Delta(t) = 1 + (t-1)(|G| + |H|) + (t-1)^2 \sum_{j \geq 0} k_j t^j,$$

where $k_j \leq I(j+1)$ for $j \geq 0$.

This gives the proof of Conjecture 1 in the case $n = 2$.

It is natural to ask whether the above theorem is valid for arbitrary $n \geq 2$. Recently, after we found our elementary combinatorial argument for $n = 2$, J. Bodnár and A. Némethi showed that the Conjecture 1 is false for $n \geq 3$, see [BN]. We provide their example below. They also found yet another proof of Conjecture 1 in the case of two singularities.

Example. Consider semigroups $S_1 = \{6k + 7l\colon k, l \geq 0\}$, $S_2 = \{2k + 9l\colon k, l \geq 0\}$, $S_3 = \{2k + 5l\colon k, l \geq 0\}$. The corresponding gap sequences are $G_1 = \{1, 2, 3, 4, 5, 8, 9, 10, 11, 15, 16, 17, 22, 23, 29\}$, $G_2 = \{1, 3, 5, 7\}$, $G_3 = \{1, 3\}$. Then,

$$\begin{aligned}\Delta_1(t) &= 1 + (t-1)(t + t^2 + t^3 + t^4 + t^5 + t^8 + t^9 + t^{10} + t^{11} \\ &\qquad + t^{15} + t^{16} + t^{17} + t^{22} + t^{23} + t^{29}) \\ \Delta_2(t) &= 1 + (t-1)(t + t^3 + t^5 + t^7) \\ \Delta_3(t) &= 1 + (t-1)(t + t^3)\end{aligned}$$

We write $\Delta = \Delta_1 \cdot \Delta_2 \cdot \Delta_3$ in the form

$$\Delta(t) = 1 + (|G_1| + |G_2| + |G_3|)(t-1) + (t-1)^2 \sum_{j \geq 0} k_j t^j.$$

One can check that

$$\begin{aligned}(k_j)_{j=0}^{\infty} = (&21, 18, 20, 15, 19, 13, 18, 11, 16, 10, 13, 10, 11, 9, 10, 7, 9, 5, 9, 3, 9, 2, \\ &7, 2, 5, 2, 4, 2, 3, 1, 3, -1, 4, -2, 4, -2, 3, -2, 2, -1, 1, 0, 0, 0, \ldots).\end{aligned}$$

We can see that $k_8 = 16$. From Theorem 1 we have $I(9) = 15$. Thus, $k_8 > I(9)$.

2 Proof of the main result

In this section we give a proof of our main result.

We begin with a simple lemma.

Lemma 2. *Take $j \geq 1$. Then the minimum of the function $J(l) = I_G(j - l) + I_H(l)$ is attained for $0 \leq l \leq j$.*

Proof. Let $l \leq 0$. Then $I_H(l) = |H| - l$ and $I_G(j-l) \geq I_G(j) + l$. Thus,

$$J(l) = I_G(j-l) + I_H(l) \geq I_G(j) + |H| = J(0).$$

In the case when $l \geq j$ we can take $l' = j - l$ and use the above inequality, exchanging the roles of G and H, to get $J(l) \geq J(j)$. □

Proof. Our goal is to express the numbers k_j in terms of the numbers k_j^G and k_j^H. We have

$$\begin{aligned}
&\Delta(t) = \Delta_G(t)\Delta_H(t) = 1 + (t-1)(|G| + |H|) \\
&+ (t-1)^2\Big[|G| \cdot |H| + \sum_{j\geq 0}(k_j^G + k_j^H)t^j + (t-1)\big(|G|\sum_{j\geq 0} k_j^H t^j + |H|\sum_{j\geq 0} k_j^G t^j\big) \\
&+ (t-1)^2\Big(\sum_{j\geq 0} k_j^G t^j\Big)\Big(\sum_{j\geq 0} k_j^H t^j\Big)\Big] = 1 + (t-1)(|G|+|H|) + (t-1)^2\Theta(t),
\end{aligned}$$

with

$$\Theta(t) = |G| \cdot |H| + k_0^G(1 - |H|) + k_0^H(1 - |G|) + k_0^G k_0^H + \sum_{j\geq 1} t^j k_j,$$

where

$$k_j = k_j^G(1 - |H|) + |H| k_{j-1}^G + k_j^H(1 - |G|) + |G| k_{j-1}^H + l_j$$

and

$$l_j = \sum_{u+v=j,\ u,v\geq 0} k_u^G k_v^H - 2\sum_{u+v=j-1,\ u,v\geq 0} k_u^G k_v^H + \sum_{u+v=j-2,\ u,v\geq 0} k_u^G k_v^H.$$

Note that $k_0^G = |G|$ and $k_0^H = |H|$. Therefore,

$$\begin{aligned}
k_0 &= |G| \cdot |H| + k_0^G(1 - |H|) + k_0^H(1 - |G|) + k_0^G k_0^H \\
&= |G| \cdot |H| + |G|(1 - |H|) + |H|(1 - |G|) + |G| \cdot |H| = |G| + |H|.
\end{aligned}$$

From Lemma 2 we get

$$I(1) = \min_{k\in\mathbb{Z}}(I_G(1-k)+I_H(k)) = \min_{k=0,1}(I_G(1-k)+I_H(k)) = |G|+|H| = k_0.$$

From now on, our goal is to prove that $k_j \le I(j+1)$ for $j \ge 1$. Note that

$$\begin{aligned} l_j = & \sum_{u+v=j,\ u,v\ge 0} k_u^G k_v^H - \sum_{u+v=j,\ u\ge 0, v\ge 1} k_u^G k_{v-1}^H - \sum_{u+v=j,\ u\ge 1, v\ge 0} k_{u-1}^G k_v^H \\ & + \sum_{u+v=j,\ u,v\ge 1} k_{u-1}^G k_{v-1}^H = \sum_{u+v=j, u,v\ge 1} (k_u^G - k_{u-1}^G)(k_v^H - k_{v-1}^H) \\ & + k_0^G k_j^H + k_j^G k_0^H - k_0^G k_{j-1}^H - k_{j-1}^G k_0^H. \end{aligned}$$

Thus,

$$k_j = \sum_{u+v=j,u,v\ge 1} (k_u^G - k_{u-1}^G)(k_v^H - k_{v-1}^H) + m_j,$$

where, miraculously,

$$\begin{aligned} m_j &= k_0^G k_j^H + k_j^G k_0^H - k_0^G k_{j-1}^H - k_{j-1}^G k_0^H \\ &\quad + k_j^G(1-|H|) + |H| k_{j-1}^G + k_j^H(1-|G|) + |G| k_{j-1}^H \\ &= |G| k_j^H + k_j^G |H| - |G| k_{j-1}^H - k_{j-1}^G |H| \\ &\quad + k_j^G(1-|H|) + |H| k_{j-1}^G + k_j^H(1-|G|) + |G| k_{j-1}^H \\ &= k_j^G + k_j^H. \end{aligned}$$

We get

$$k_j = k_j^G + k_j^H + \sum_{u+v=j,u,v\ge 1} (k_{u-1}^G - k_u^G)(k_{v-1}^H - k_v^H).$$

We are to prove that $k_j \le (I_G \diamond I_H)(j+1)$. It suffices to prove that $k_j \le I_G(j+1-l) + I_H(l)$ for every $l \in \mathbb{Z}$. Thus, we have to deal with the inequality

$$k_j^G + k_j^H + \sum_{u+v=j,u,v\ge 1} (k_u^G - k_{u-1}^G)(k_v^H - k_{v-1}^H) \le I_G(j+1-l) + I_H(l), \quad j \ge 1, l \in \mathbb{Z}.$$

By Lemma 2 it suffices to consider $0 \le l \le j+1$. Note that if $u+v=j$ then we either have $u \ge j-l+1$ or $v \ge l$. Thus,

$$\mathbf{1}_{u\in G}\mathbf{1}_{v\in H}\mathbf{1}_{u+v=j} \le \mathbf{1}_{u\in G\cap[j-l+1,j]} + \mathbf{1}_{v\in H\cap[l,j]},$$

where the indicator functions in the above expression are functions of two variables u and v. We have also used the convention $[a, b] = \emptyset$ for $a > b$. We obtain

$$\begin{aligned}\sum_{u+v=j,u,v\geq 1} (k^G_{u-1} - k^G_u)(k^H_{v-1} - k^H_v) &= \sum_{u+v=j,u,v\geq 0} (k^G_{u-1} - k^G_u)(k^H_{v-1} - k^H_v) \\ &= \sum_{u+v=j,u,v\geq 0} \mathbf{1}_{u\in G}\mathbf{1}_{v\in H} \leq \sum_{u+v=j,u,v\geq 0} \left(\mathbf{1}_{u\in G\cap[j-l+1,j]} + \mathbf{1}_{v\in H\cap[l,j]}\right) \\ &= (I_G(j-l+1) - I_G(j+1)) + (I_H(l) - I_H(j+1)) = \\ &- (k^G_j + k^H_j) + I_G(j+1-l) + I_H(l).\end{aligned}$$

This concludes the proof. □

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