On polynomially bounded harmonic functions on the $\mathbb{Z}^d$ lattice

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Abstract

We prove that if $f : \mathbb{Z}^d \to \mathbb{R}$ is harmonic and there exists a polynomial $W : \mathbb{Z}^d \to \mathbb{R}$ such that $f + W$ is nonnegative, then f is a polynomial.

1 Introduction

Harmonic functions on the integer lattice are closely related to lattice random walks and have been studied by many authors - some introduction and detailed references can be found in a modern monographic book by Woess, [8]. Many different methods have been succesfully applied, including the extreme point theory, [2], and martingale approach, [4]. The present paper grew out of the author's licentiate thesis, [7] which extended results and methods of Darkiewicz, [3]. Similar result for sublinear functions on compactly generated groups having polynomial growth has been obtained in a paper by Hebisch and Saloff-Coste, [6] (Theorem 6.1), by using Gaussian estimates for iterated kernels of random walks.

2 Preliminaries and main results

Let $d \in \mathbb{N}$ and let $(e_i)_{i=1}^d$ be the standard orthonormal basis for $\mathbb{R}^d$. The function $f : \mathbb{Z}^d \to \mathbb{R}$ is called harmonic if it has the mean value property

$$f(x) = \frac{1}{2d} \sum_{i=1}^{d} [f(x+e_i) + f(x-e_i)] \qquad \text{for all } x \in \mathbb{Z}^d.$$

2000 *Mathematics Subject Classification*: Primary 31C05; Secondary 60G50.
Key words and phrases: harmonic functions, integer lattice, polynomially bounded harmonic functions

We say that $f : \mathbb{Z}^d \to \mathbb{R}$ is a polynomial if there exists a polynomial $F : \mathbb{R}^d \to \mathbb{R}$ such that $f = F|_{\mathbb{Z}^d}$.

For $t \geq 0$ let $Y_1^{(t)}, \dots, Y_d^{(t)}, Z_1^{(t)}, \dots, Z_d^{(t)}$ be independent Poisson random variables with mean t.

We will use the following notation:

- $\|x\|_p = (\sum_{i=1}^d |x_i|^p)^{1/p}$ for $p \in [1, \infty)$ and $x = (x_1, \dots, x_d) \in \mathbb{R}^d$
- $X_i^{(t)} = Y_i^{(t)} - Z_i^{(t)}$ for $i = 1, \dots, d$, $\quad X^{(t)} = \sum_{i=1}^d X_i^{(t)} e_i$
- $g_t(l) = \mathbb{P}(Y_1^{(t)} - Z_1^{(t)} = l)$ for $l \in \mathbb{Z}$
- $G_t(k) = \prod_{i=1}^m g_t(k_i)$ for $k = (k_1, \dots, k_m) \in \mathbb{Z}^m$
- $q_t(l) = \mathbb{P}(Y_1^{(t)} = l) = e^{-t} t^l / l!$ for $l \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$

Note that if $t \in \mathbb{N}$ then $q_t(0) \leq q_t(1) \leq \ldots \leq q_t(t-1) = q_t(t) \geq q_t(t+1) \geq q_t(t+2) \geq \ldots$

We consider the space of all exponentially bounded functions

$$\mathcal{L} = \{ f : \mathbb{Z}^d \to \mathbb{R} \quad | \quad \exists_{c_1, c_2 > 0} \quad |f(x)| \leq c_1 e^{c_2 \|x\|_1} \quad \text{for all } x \in \mathbb{Z}^d \}$$

and define a family of operators $(\mathcal{P}_t)_{t \geq 0}$, $\mathcal{P}_t : \mathcal{L} \to \mathcal{L}$ by

$$\mathcal{P}_t(f)(x) = \mathbb{E} f\big(x + X^{(t)}\big).$$

Theorem 2.1. *The family $(\mathcal{P}_t)_{t \geq 0}$ is a well-defined semigroup of operators. Moreover, harmonic functions belonging to $\mathcal{L}$ lie in a domain $\mathcal{D}_A$ of an infinitesimal generator A of the semigroup $(\mathcal{P}_t)_{t \geq 0}$ and for $f \in \mathcal{D}_A$ we have*

$$(Af)(x) = \frac{d}{dt} \mathcal{P}_t(f)(x)\big|_{t=0} = \sum_{k \in \mathbb{Z}^d : \|k\|_1 = 1} f(x+k) - 2d f(x).$$

In particular, if $f \in \mathcal{L}$ is harmonic, then for all $x \in \mathbb{Z}^d$ there is $(Af)(x) = 0$ and so for $x \in \mathbb{Z}^d$

$$\mathcal{P}_t(f)(x) = \sum_{k \in \mathbb{Z}^d} G_t(k) f(x+k) = f(x).$$

Proof. If $f \in \mathcal{L}$, then there exist $c_1, c_2, \tilde{c}_1(t) > 0$ such that

$$|\mathbb{E} f(x + X^{(t)})| \leq c_1 \mathbb{E} e^{c_2 \|x + X^{(t)}\|_1} \leq c_1 e^{c_2 \|x\|_1} (\mathbb{E} e^{c_2 |X_1^{(t)}|})^d = \tilde{c}_1(t) e^{c_2 \|x\|_1},$$

so $\mathcal{P}_t(f) \in \mathcal{L}$. Observe that $\mathcal{P}_0(f) = f$. If $s, t \geq 0$ and $\tilde{X}^{(s)}$ is a copy of $X^{(s)}$ independent of $X^{(t)}$, then $X^{(t)} + \tilde{X}^{(s)} \sim X^{(t+s)}$, so one can easily check that $(\mathcal{P}_t)_{t \geq 0}$ is a semigroup. The last part is a simple calculation. $\square$

Lemma 2.2. *If $(r_i)_{i\in\mathbb{N}}$ are independent ±1 symmetric Bernoulli random variables and M is a Poisson variable with mean $4t$, such that M, $(r_i)_{i\in\mathbb{N}}$ are independent, then*

$$X_1^{(t)} \sim \frac{1}{2}\left(r_1 + \ldots + r_{2M}\right).$$

Moreover, for $l \in \mathbb{N}_0$

$$g_t(l) = g_t(-l) = \sum_{n=0}^{\infty} e^{-4t}\frac{t^n}{n!}\binom{2n}{n+l},$$

so if $0 \le l_1 \le l_2$; $l_1, l_2 \in \mathbb{Z}$, then

$$g_t(l_1) \ge g_t(l_2).$$

Proof. It is enough to show that the characteristic functions of both random variables are equal. We have

$$\phi_{X_1^{(t)}}(x) = \phi_{Y_1^{(t)}}(x)\phi_{Z_1^{(t)}}(-x) = e^{t(e^{ix}-1)}e^{t(e^{-ix}-1)} = e^{t(2\cos x - 2)} = e^{-4t\sin^2(x/2)}$$

and

$$\begin{aligned}\phi_{(r_1+\ldots+r_{2M})/2}(x) &= \sum_{n=0}^{\infty}\mathbb{P}\left(M=n\right)\phi_{(r_1+\ldots+r_{2n})/2}(x) = \sum_{n=0}^{\infty} e^{-4t}\frac{(4t)^n}{n!}\left(\phi_{r_1/2}(x)\right)^{2n} \\ &= e^{-4t}e^{4t(\phi_{r_1/2}(x))^2} = e^{4t(-1+\cos^2(x/2))} = e^{-4t\sin^2(x/2)},\end{aligned}$$

as

$$\phi_{r_1/2}(x) = \phi_{r_1}(x/2) = \frac{1}{2}\left(e^{-ix/2} + e^{ix/2}\right) = \cos(x/2).$$

To finish the proof observe that for $l \in \mathbb{N}_0$ we have

$$\begin{aligned}g_t(l) &= \mathbb{P}\left(\frac{1}{2}(r_1 + \ldots + r_{2M}) = l\right) = \sum_{n=0}^{\infty}\mathbb{P}\left(M=n\right)\mathbb{P}\left(r_1 + \ldots + r_{2n} = 2l\right) \\ &= \sum_{n=0}^{\infty} e^{-4t}\frac{(4t)^n}{n!}\frac{1}{2^{2n}}\binom{2n}{n+l} = \sum_{n=0}^{\infty} e^{-4t}\frac{t^n}{n!}\binom{2n}{n+l}\end{aligned}$$

and $\binom{2n}{n+l_1} \ge \binom{2n}{n+l_2}$ for $0 \le l_1 \le l_2$. □

Lemma 2.3. *For $\varepsilon > 0$ and $d \in \mathbb{N}$ we can find $0 < s < t$ such that*

$$g_t(k) \ge (1-\varepsilon)g_s(k-1) \qquad \text{for } k \in \mathbb{Z}$$

and

$$G_t(k) \ge (1-\varepsilon)G_s(k-e_1) \qquad \text{for } k \in \mathbb{Z}^d.$$

Proof. If the first inequality holds for $k = 1, 2, \ldots, m$ then it holds for $k = 0, -1, \ldots, -m$. Indeed, for $k = -1, -2, \ldots, -m$ we have (see Lemma 2.2)

$$\begin{aligned}\mathbb{P}\left(X_1^{(t)} = k\right) &= \mathbb{P}\left(X_1^{(t)} = -k\right) \ge (1-\varepsilon)\mathbb{P}\left(X_1^{(s)} = -k-1\right) \\ &= (1-\varepsilon)\mathbb{P}\left(X_1^{(s)} = k+1\right) \ge (1-\varepsilon)\mathbb{P}\left(X_1^{(s)} = k-1\right)\end{aligned}$$

and

$$\mathbb{P}\left(X_1^{(t)}=0\right)\geq\mathbb{P}\left(X_1^{(t)}=1\right)\geq(1-\varepsilon)\mathbb{P}\left(X_1^{(s)}=0\right)\geq(1-\varepsilon)\mathbb{P}\left(X_1^{(s)}=-1\right).$$

For $k\geq 1$ we have

$$\mathbb{P}\left(X_t=k\right)=\sum_{l=0}^{\infty}\mathbb{P}\left(Y_t=l+k\right)\mathbb{P}\left(Z_t=l\right)=\sum_{l=0}^{\infty}e^{-2t}\frac{t^{2l+k}}{l!(l+k)!},$$

$$\mathbb{P}\left(X_s=k-1\right)=\sum_{l=0}^{\infty}e^{-2s}\frac{s^{2l+k-1}}{l!(l+k-1)!}.$$

Let $s>1$ be such that $\sqrt{s}\in\mathbb{N}$ and set $t=s+\sqrt{s}$. We then have

$$\mathbb{P}\left(X_t=k\right)\geq\sum_{l=\sqrt{s}}^{\infty}e^{-2t}\frac{t^{2l+k}}{l!(l+k)!}=\sum_{l=0}^{\infty}e^{-2t}\frac{t^{2(l+\sqrt{s})+k}}{(l+\sqrt{s})!(l+\sqrt{s}+k)!}.$$

It is enough to prove that

$$\inf_{k\geq 1,l\geq 0}\left(e^{-2t}\frac{t^{2(l+\sqrt{s})+k}}{(l+\sqrt{s})!(l+\sqrt{s}+k)!}\;\Big/\;e^{-2s}\frac{s^{2l+k-1}}{l!(l+k-1)!}\right)\xrightarrow[s\to\infty]{}1.$$

We then consider the expression

$$p_{l,k}(s,t(s)):=e^{2(s-t)}s\,t^{2\sqrt{s}}\left(\frac{t}{s}\right)^{l+k}\frac{(l+k-1)!}{(l+\sqrt{s}+k)!}\left(\frac{t}{s}\right)^{l}\frac{l!}{(l+\sqrt{s})!}.$$

A function $\mathbb{N}\ni n\mapsto(t/s)^n(n-1)!/(n+\sqrt{s})!$ has its minimum at $n=s(1+\sqrt{s})/(t-s)=t$. Similarly, a function $\mathbb{N}_0\ni n\mapsto(t/s)^n n!/(n+\sqrt{s})!$ has its minimum at $n=s\sqrt{s}/(t-s)=s$. Therefore

$$\begin{aligned}p_{l,k}(s,t(s))\geq p_{s,t-s}(s,t(s))&=e^{2(s-t)}s\,t^{2\sqrt{s}}\left(\frac{t}{s}\right)^{t+s}\frac{(t-1)!}{(t+\sqrt{s})!}\frac{s!}{t!}\\&=e^{-2\sqrt{s}}s(s+\sqrt{s})^{2\sqrt{s}}\left(\frac{s+\sqrt{s}}{s}\right)^{2s+\sqrt{s}}\frac{s!}{(s+2\sqrt{s})!}\frac{1}{s+\sqrt{s}}.\end{aligned}$$

Using Stirling's formula we get $s!/(s+2\sqrt{s})!\approx e^{2\sqrt{s}}s^s/(s+2\sqrt{s})^{s+2\sqrt{s}}$ as $s\to\infty$, hence we arrive at

$$\begin{aligned}\inf_{k\geq 1,l\geq 0}p_{l,k}(s)&\approx s^{-s-\sqrt{s}+1}(s+\sqrt{s})^{2s+3\sqrt{s}-1}(s+2\sqrt{s})^{-s-2\sqrt{s}}\\&=\sqrt{s}^{-2s-2\sqrt{s}+2+2s+3\sqrt{s}-1}(1+\sqrt{s})^{-\sqrt{s}-1}(1+\sqrt{s})^{2s+4\sqrt{s}}(s+2\sqrt{s})^{-s-2\sqrt{s}}\\&=\left(\frac{\sqrt{s}}{1+\sqrt{s}}\right)^{\sqrt{s}+1}\left(\frac{s+2\sqrt{s}+1}{s+2\sqrt{s}}\right)^{s+2\sqrt{s}}\xrightarrow[s\to\infty]{}e^{-1}e=1.\end{aligned}$$

To prove the second part observe that the first inequality yields

$$G_t(k)=g_t(k_1)\cdot\ldots\cdot g_t(k_d)\geq(1-\varepsilon)g_s(k_1-1)g_t(k_2)\cdot\ldots\cdot g_t(k_d)\geq(1-\varepsilon)^d G_s(k-e_1),$$

since

$$g_t(l) = g_t(|l|) \geq g_t(|l|+1) \geq (1-\varepsilon) g_s(|l|) = (1-\varepsilon) g_s(l). \quad \square$$

A sequence $(x_i)_{i=0}^n \subset \mathbb{Z}^d$ is called a *path* in $\mathbb{Z}^d$ between x_0 and x_n if $\|x_i - x_{i+1}\|_1 = 1$ for $i = 0, \ldots, n-1$. For $k \in \mathbb{Z}^d$ let $L_n(k)$ denote the number of paths in $\mathbb{Z}^d$ between 0 and k.

Lemma 2.4. *Let $f : \mathbb{Z}^d \to \mathbb{R}$ be harmonic. Suppose there exists a polynomial $W : \mathbb{Z}^d \to \mathbb{R}$ such that $f(x) \geq -W(x)$. Then $f \in \mathcal{L}$.*

Proof. Using simple induction we prove that for f harmonic and $n \in \mathbb{N}$ we have

$$f(0) = \frac{1}{(2d)^n} \sum_{k \in \mathbb{Z}^d} f(k) L_n(k).$$

Let $l \in \mathbb{Z}^d$. Then $L_{\|l\|_1}(l) \geq 1$ and

$$\begin{aligned} f(0)(2d)^{\|l\|_1} &= \sum_{k \in \mathbb{Z}^d} (f(k) + W(k)) L_{\|l\|_1}(k) - \sum_{k \in \mathbb{Z}^d} W(k) L_{\|l\|_1}(k) \\ &\geq (f(l) + W(l)) - \max_{k : \|k\|_1 \leq \|l\|_1} |W(k)| \cdot (2d)^{\|l\|_1}, \end{aligned}$$

hence

$$f(l) \leq f(0)(2d)^{\|l\|_1} + (2d)^{\|l\|_1} \cdot \max_{k : \|k\|_1 \leq \|l\|_1} |W(k)| - W(l) \leq c_1 e^{c_2 \|l\|_1}$$

for some $c_1, c_2 > 0$ which depend only on f and W but not on l. Since f is polynomially bounded from below we have $f \in \mathcal{L}$. $\square$

Now we may recover the classical strong Liouvillle property of harmonic functions on $\mathbb{Z}^d$. Woess, [8], traces back its weak form to Blackwell, [1]; see also [2] and [5].

Theorem 2.5. *If $f : \mathbb{Z}^d \to \mathbb{R}$ is harmonic and $f \geq 0$ then f is constant.*

Proof. By Lemma 2.4 we have $f \in \mathcal{L}$. Let $x \in \mathbb{Z}^d$. Lemma 2.3 implies that there exist $t > s > 0$ such that

$$\begin{aligned} f(x) - f(x+e_1) = P_t(f)(x) - P_s(f)(x+e_1) &= \sum_{k \in \mathbb{Z}^d} f(x+k) G_t(k) - \sum_{k \in \mathbb{Z}^d} f(x+k+e_1) G_s(k) \\ &= \sum_{k \in \mathbb{Z}^d} f(x+k) \left(G_t(k) - G_s(k-e_1)\right) \\ &\geq -\varepsilon \sum_{k \in \mathbb{Z}^d} f(x+k) G_s(k-e_1) = -\varepsilon f(x+e_1), \end{aligned}$$

By taking $\varepsilon \to 0$ we get $f(x) \geq f(x+e_1)$. Applying this inequality to the harmonic function $x \mapsto g(x) = f(-x)$ we get $f(x) = f(x+e_1)$ and similarly $f(x) = f(x+e_i)$ for $i = 1, \ldots, d$. $\square$

We will now prove some auxiliary lemmas.

Lemma 2.6. *Let $n \in \mathbb{N}$ and let $k \in \mathbb{Z}$ satisfy $|k| \le n$. Then*

$$\frac{1}{2\sqrt{n}}\left(1-\frac{k^2}{n}\right) \le \frac{1}{2^{2n}}\binom{2n}{n+k} \le \frac{1}{\sqrt{2n+1}}e^{-\frac{k^2}{2n}} \le \frac{1}{\sqrt{n+1}}e^{-\frac{k^2}{2n}}.$$

Proof. We can assume $k \ge 0$. By multiplying the obvious inequalities $(2j-1)^2 \ge 2j(2j-2)$ for $j = 2, 3, \dots, n$ and $(2j)^2 \ge (2j-1)(2j+1)$ for $j = 1, 2, \dots, n$ we arrive at $((2n-1)!!)^2 \ge \frac{1}{2}(2n)!!(2n-2)!!$ and $((2n)!!)^2 \ge (2n-1)!!(2n+1)!!$, so that

$$\frac{1}{4n} \le \left(\frac{(2n-1)!!}{(2n)!!}\right)^2 \le \frac{1}{2n+1}.$$

To finish the proof it suffices to observe that

$$\frac{1}{2^{2n}}\binom{2n}{n+k} = \frac{(2n-1)!!}{(2n)!!}\cdot\prod_{j=1}^{k}\left(1-\frac{k}{n+j}\right)$$

and

$$1-\frac{k^2}{n} \le \left(1-\frac{k}{n}\right)^k \le \prod_{j=1}^{k}\left(1-\frac{k}{n+j}\right) \le \left(1-\frac{k}{2n}\right)^k \le e^{-\frac{k^2}{2n}}. \quad \square$$

Lemma 2.7. *There exists a constant $C > 0$ such that for $k \in \mathbb{Z}^d \setminus \{0\}$*

$$G_{\|k\|_1^2}(k) \ge C^d \cdot \|k\|_1^{-2d} \ .$$

Proof. Let $t > 0$ and $k = (k_1, \dots, k_d) \in \mathbb{Z}^d$. We have (see Lemma 2.2)

$$g_t(k_i) \ge e^{-4t}\frac{t^n}{n!}\binom{2n}{n+k_i} \ge e^{-4t}\frac{t^n}{n!}\binom{2n}{n+\|k\|_1} \qquad (i = 1, \dots, d, \quad n \in \mathbb{N}).$$

We set $t = \|k\|_1^2$ and $n = 4t$. Then $e^{-4t}t^n = e^{-n}n^n/4^n$, so that

$$g_t(k_i) \ge q_n(n)\cdot\frac{1}{2^{2n}}\binom{2n}{n+\|k\|_1} \ge q_n(n)\cdot\frac{1}{2\sqrt{n}}\left(1-\frac{\|k\|_1^2}{n}\right) = \frac{3}{16}q_n(n)/\|k\|_1,$$

where we have used Lemma 2.6. Note that by Chebyshev's inequality

$$\mathbb{P}(|Y_1^{(n)} - n| \ge 2\sqrt{n}) = \mathbb{P}(|Y_1^{(n)} - \mathbb{E}Y_1^{(n)}| \ge 2\sqrt{n}) \le \frac{D^2Y_1^{(n)}}{4n} = 1/4,$$

so that

$$\begin{aligned} 3/4 \le \mathbb{P}(|Y_1^{(n)} - n| < 2\sqrt{n}) &= \sum_{m\in\mathbb{N}_0:|m-n|<2\sqrt{n}} q_n(m) \\ &\le card\{m \in \mathbb{N}_0 : |m-n| < 2\sqrt{n}\}\cdot q_n(n) \le 8\|k\|_1\cdot q_n(n). \end{aligned}$$

Hence

$$g_t(k_i) \ge \frac{3}{32\|k\|_1}\cdot\frac{3}{16\|k\|_1} = \frac{C}{\|k\|_1^2}$$

and therefore

$$G_{\|k\|_1^2}(k) = \prod_{i=1}^{d} g_t(k_i) \ge C^d\cdot\|k\|_1^{-2d}. \ \square$$

Lemma 2.8. *Let $W : \mathbb{R}^d \to \mathbb{R}$ be a polynomial. We define $H_W : \mathbb{R} \to \mathbb{R}$ by*

$$H_W(t) = \mathcal{P}_t(W)(0) = \sum_{k\in\mathbb{Z}^d} G_t(k)W(k).$$

Then H_W is a polynomial.

Proof. H_W is well-defined since $W\big|_{\mathbb{Z}^d} \in \mathcal{L}$. Because of the product structure of G_t it is enough to consider the case $d = 1$ and $W(z) = z^l$ for $l \in \mathbb{N}$. The characteristic function

$$\phi_{X_1^{(t)}}(z) = e^{-4t\sin^2(z/2)}$$

is smooth, so that

$$H_W(t) = \mathbb{E}[(X_1^{(t)})^l] = (-i)^l \frac{\mathrm{d}^l \phi_{X_1^{(t)}}}{\mathrm{d}z^l}(0)$$

which clearly is a polynomial in variable t. □

Lemma 2.9. *Let $f : \mathbb{Z}^d \to \mathbb{R}$ be harmonic. Suppose there exists a polynomial $W : \mathbb{Z}^d \to \mathbb{R}$ such that $f \geq -W$. Then $|f| \leq R$ for some polynomial $R : \mathbb{Z}^d \to \mathbb{R}$.*

Proof. We have $f \in \mathcal{L}$ (see Lemma 2.4). Proposition 2.1 yields

$$f(0) = \sum_{k\in\mathbb{Z}^d} G_t(k)f(k),$$

hence for $l \in \mathbb{Z}^d$

$$\begin{aligned} f(0) &= \sum_{k\in\mathbb{Z}^d} G_t(k)\left(f(k) + W(k)\right) - \sum_{k\in\mathbb{Z}^d} G_t(k)W(k) \\ &\geq G_t(l)(f(l) + W(l)) - H_W(t). \end{aligned}$$

Therefore

$$f(0) + H_W(t) \geq G_t(l)(f(l) + W(l)).$$

There exists a constant $c = c(d) > 0$ such that (see Lemma 2.7) for $l \neq 0$

$$G_{\|l\|_1^2}(l) \geq c \cdot \|l\|_1^{-2d}.$$

Hence for $l \neq 0$

$$f(0) + H_W(\|l\|_1^2) \geq c \cdot (f(l) + W(l)) \cdot \|l\|_1^{-2d}$$

and therefore

$$f(l) \leq c^{-1}\|l\|_1^{2d}\left(f(0) + H_W(\|l\|_1^2)\right) - W(l).$$

Since the right-hand side of the above inequality is polynomially bounded from above in variable l, we have $f(l) \leq P(l)$ for some polynomial $P : \mathbb{R}^d \to \mathbb{R}$ and for all $l \in \mathbb{Z}^d$. One can easily check that $|f(l)| \leq 1 + [P(l)]^2 + [W(l)]^2$. □

Lemma 2.10. *For $x \in \mathbb{Z}$, $n \in \mathbb{N}$, $a, b \in \mathbb{R}$ and $p \geq 0$ we have*

$$|a+b|^p \leq 2^p(|a|^p + |b|^p)$$

and

$$||x|^n - |x+1|^n| \leq 1 + 2^n|x|^{n-1}.$$

Proof. Without loss of generality we may assume that $|a| \leq |b|$. Then

$$|a+b|^p \leq (2|b|)^p \leq 2^p(|a|^p + |b|^p).$$

To prove the second inequality note that

$$||(x+1)^n| - |x^n|| \leq |(x+1)^n - x^n| = \Big|\sum_{k=0}^{n-1}\binom{n}{k}x^k\Big| \leq 1 + \sum_{k=1}^{n-1}\binom{n}{k}|x|^{n-1} \leq 1 + 2^n|x|^{n-1}. \square$$

Lemma 2.11. *If $t > 0$ then*

$$g_t(0) \leq \frac{1}{2\sqrt{t}}$$

and

$$\mathbb{E}|X_1^{(t)}|^m \leq b(m)t^{m/2} + c(m)$$

for some constants $b(m), c(m) > 0$ and $m \in \mathbb{N}$.

Proof. Let M be the Poisson variable with mean $4t$. By Lemma 2.2, Lemma 2.6 and Jensen's inequality we have

$$g_t(0) = \sum_{n=0}^{\infty} e^{-4t}\frac{(4t)^n}{n!}\frac{1}{2^{2n}}\binom{2n}{n} \leq \sum_{n=0}^{\infty} e^{-4t}\frac{(4t)^n}{n!}\frac{1}{\sqrt{n+1}} = \mathbb{E}\frac{1}{\sqrt{M+1}} \leq \left(\mathbb{E}\frac{1}{M+1}\right)^{1/2}$$

and

$$\mathbb{E}\frac{1}{M+1} = \sum_{n=0}^{\infty} e^{-4t}\frac{(4t)^n}{(n+1)!} = \frac{1}{4t}\sum_{n=0}^{\infty} e^{-4t}\frac{(4t)^{n+1}}{(n+1)!} \leq \frac{1}{4t}.$$

Now let us prove the second part. Let M, r_1, $r_2, \ldots$ be as in Lemma 2.2. For fixed $k \in \mathbb{N}$ and $i \leq k$ we have $\mathbb{E}e^{r_i/\sqrt{k}} = 1 + \sum_{s=1}^{\infty} k^{-s}/(2s)! \leq 1 + ek^{-1} \leq e^{e/k}$, so that

$$\frac{1}{m!}\mathbb{E}\left(\frac{r_1 + r_2 + \ldots + r_k}{\sqrt{k}}\right)_+^m \leq \mathbb{E}\exp\left(\frac{r_1 + \ldots + r_k}{\sqrt{k}}\right) = \prod_{i=1}^{k}\mathbb{E}e^{r_i/\sqrt{k}} \leq e^e.$$

Hence

$$\mathbb{E}|r_1 + \ldots + r_k|^m = 2\mathbb{E}(r_1 + \ldots + r_k)_+^m \leq 2e^e m! \cdot k^{m/2}$$

and therefore, by Lemma 2.2,

$$\mathbb{E}|X_1^{(t)}|^m \leq 2e^e m! \cdot 2^{-m} \cdot \mathbb{E}(2M)^{m/2} \leq 2e^e m! \cdot (\mathbb{E}M^m)^{1/2}.$$

Now,

$$\begin{aligned}\mathbb{E}M^m &= \mathbb{E}M^m I_{M<m} + \mathbb{E}M^m I_{M\geq m} \leq m^m + m^m \mathbb{E}(M-m+1)^m \\ &\leq m^m \left(1 + \sum_{k=m}^{\infty} e^{-4t}\frac{(4t)^k}{k!} k(k-1)\dots(k-m+1)\right) = m^m(1+(4t)^m)\end{aligned}$$

and it is obvious (see Lemma 2.10) that

$$\mathbb{E}|X_1^{(t)}|^m \leq b(m)t^{m/2} + c(m)$$

for some constants $b(m), c(m) > 0$.

Now we state the key lemma of this paper. Similar estimates for sublinear harmonic functions has been obtained in a more general setting in the paper [6] (Theorem 6.1) by using Theorem 5.1 (inequality (14)).

Lemma 2.12. *Let $n \in \mathbb{N}$ and let $f : \mathbb{Z}^d \to \mathbb{R}$ be harmonic. Suppose that there exists a constant a_n such that*

$$|f(x)| \leq a_n(1 + \|x\|_n^n)$$

for all $x \in \mathbb{Z}^d$. Then there exists a constant a_{n-1} such that for all $x \in \mathbb{Z}^d$

$$|f(x+e_1) - f(x)| \leq a_{n-1}(1 + \|x\|_{n-1}^{n-1}).$$

Proof. For $x \in \mathbb{Z}^d$ and any $t > 0$ we have

$$f(x) = \sum_{k\in\mathbb{Z}^d} G_t(k) f(x+k)$$

and

$$f(x+e_1) = \sum_{k\in\mathbb{Z}^d} G_t(k) f(x+e_1+k) = \sum_{k\in\mathbb{Z}^d} G_t(k-e_1) f(x+k),$$

hence

$$\begin{aligned}|f(x+e_1) - f(x)| &\leq \sum_{k\in\mathbb{Z}^d} |G_t(k-e_1) - G_t(k)||f(x+k)| \\ &\leq \sum_{k\in\mathbb{Z}^d} |G_t(k-e_1) - G_t(k)| a_n(1+\|x+k\|_n^n)\end{aligned}$$

$$\begin{aligned}&= \sum_{k\in\mathbb{Z}^d:k_1\leq 0} (G_t(k) - G_t(k-e_1))a_n(1+\|x+k\|_n^n) + \sum_{k\in\mathbb{Z}^d:k_1>0} (G_t(k-e_1) - G_t(k))a_n(1+\|x+k\|_n^n) \\ &= \sum_{k\in\mathbb{Z}^d:k_1\leq -1} G_t(k)a_n(|x_1+k_1|^n - |x_1+k_1+1|^n) + \sum_{k\in\mathbb{Z}^d:k_1\geq 1} G_t(k)a_n(|x_1+k_1+1|^n - |x_1+k_1|^n) \\ &\quad + \sum_{k\in\{0\}\times\mathbb{Z}^{d-1}} G_t(k)a_n(1+\|x+k\|_n^n) + \sum_{k\in\{0\}\times\mathbb{Z}^{d-1}} G_t(k)a_n(1+\|x+k+e_1\|_n^n).\end{aligned}$$

We have used the product structure of G_t and Lemma 2.2. By using Lemma 2.10 we get

$$\sum_{k_1\le -1, k\in\mathbb{Z}^d} G_t(k)(|x_1+k_1|^n - |x_1+k_1+1|^n) + \sum_{k_1\ge 1, k\in\mathbb{Z}^d} G_t(k)(|x_1+k_1+1|^n - |x_1+k_1|^n)$$

$$\begin{aligned}
&\le \sum_{k\in\mathbb{Z}^d} G_t(k)(2^n|x_1+k_1|^{n-1}+1) = 1 + 2^n \sum_{k_1\in\mathbb{Z}} g_t(k_1)|x_1+k_1|^{n-1} \\
&\le 1 + 2^{2n-1}\sum_{k_1\in\mathbb{Z}} g_t(k_1)(|x_1|^{n-1}+|k_1|^{n-1}) = 1+2^{2n-1}\left(|x_1|^{n-1} + \mathbb{E}|X_1^{(t)}|^{n-1}\right)
\end{aligned}$$

We also have, again by using Lemma 2.10 several times,

$$\sum_{k\in\{0\}\times\mathbb{Z}^{d-1}} G_t(k)(1+\|x+k\|_n^n) + \sum_{k\in\{0\}\times\mathbb{Z}^{d-1}} G_t(k)(1+\|x+k+e_1\|_n^n)$$

$$\begin{aligned}
&\le \sum_{k\in\{0\}\times\mathbb{Z}^{d-1}} G_t(k)\left(2+2^n\|x\|_n^n + 2^n\|x+e_1\|_n^n + 2^{n+1}\|k\|_n^n\right) \\
&\le g_t(0)\left(2+2^n\|x\|_n^n + 2^n\|x+e_1\|_n^n + d\,2^{n+1}\mathbb{E}|X_1^{(t)}|^n\right) \\
&\le 4^{n+1}g_t(0)\left(1+\|x\|_n^n + d\,\mathbb{E}|X_1^{(t)}|^n\right),
\end{aligned}$$

so we arrive at

$$\begin{aligned}
|f(x+e_1)-f(x)| &\le a_n\left(1+2^{2n-1}\left(|x_1|^{n-1}+\mathbb{E}|X_1^{(t)}|^{n-1}\right) + 4^{n+1}g_t(0)\left(1+\|x\|_n^n + d\,\mathbb{E}|X_1^{(t)}|^n\right)\right) \\
&\le 4^{n+2}a_n d\left[\left(1+\|x\|_{n-1}^{n-1}+\mathbb{E}|X_1^{(t)}|^{n-1}\right) + g_t(0)\left(\|x\|_n^n + \mathbb{E}|X_1^{(t)}|^n\right)\right].
\end{aligned}$$

From Lemma 2.11 we infer that there exists a constant $C = C(n,d)$ such that for every $t>0$ and every $x\in\mathbb{Z}^d$ there is

$$|f(x+e_1)-f(x)| \le Ca_n\left[1+\|x\|_{n-1}^{n-1} + t^{\frac{n-1}{2}} + t^{-1/2}\left(\|x\|_n^n + t^{n/2}\right)\right].$$

By setting $t=(1+\|x\|_1)^2$ we complete the proof. □

Lemma 2.13. *Let $f:\mathbb{Z}^d\to\mathbb{R}$ be such that $f_i(x) = f(x+e_i)-f(x)$ are polynomials for $i=1,2,\dots,d$. Then f is a polynomial.*

Proof. To begin with we consider the case $d=1$. Note that $f(x)-f(0)$ is determined by values of f_1. Define a sequence of polynomials $(W_k)_{k=0}^\infty$ by

$$x^m = \sum_{k=0}^{m-1}\binom{m}{k}W_k(x); \quad m=1,2,\dots$$

A simple induction yields that $W_k(x+1) - W_k(x) = x^k$ and $W_k(0) = 0$. It follows that if $f_1(x) = \sum_{i=0}^{l} a_i x^i$ then $f(x) = f(0) + \sum_{i=0}^{l} a_i W_i(x)$. If $d > 1$ then

$$f(x_1, \ldots, x_d) = f(x_1, x_2, \ldots, x_d) - f(0, x_2, \ldots, x_d) + f(0, x_2, \ldots, x_d) - f(0, 0, x_3, \ldots, x_d) \\ + \ldots + f(0, \ldots, 0, x_1) - f(0, \ldots, 0) + f(0).$$

By using the same argument as in the case $d = 1$ we see that

$$f(0, \ldots, x_i, \ldots, x_d) - f(0, \ldots, x_{i+1}, \ldots, x_d), \qquad (i = 1, \ldots, d)$$

are polynomials. □

Main Theorem 2.14. *Let $f : \mathbb{Z}^d \to \mathbb{R}$ be harmonic. Suppose there exists a polynomial $W : \mathbb{Z}^d \to \mathbb{R}$ such that $f(k) \geq -W(k)$ for $k \in \mathbb{Z}^d$. Then f is a polynomial.*

Proof. There exists (see Lemma 2.9) $n \in \mathbb{N}$ such that $|f(x)| \leq a_n(1 + \|x\|_n^n)$. We claim that together with the harmonicity of f this already implies that f is a polynomial. We prove this claim by induction with respect to the parameter n. For $n = 0$ the claim is a consequence of Proposition 2.5. For $n > 1$ let $f_i(x) = f_i(x + e_1) - f(x)$. Note that $f_i, i = 1, \ldots, d$ are also harmonic. By the Lemma 2.12 and induction hypothesis, f_i are polynomials, hence by Lemma 2.13 we get that f is a polynomial as well. □

Acknowledgements

I would like to thank prof. Krzysztof Oleszkiewicz for his very helpful comments and suggestions, especially for giving me ideas of Proposition 2.1 and Lemma 2.2.

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