ON A LOOMIS-WHITNEY TYPE INEQUALITY FOR PERMUTATIONALLY INVARIANT UNCONDITIONAL CONVEX BODIES

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Abstract. For a permutationally invariant unconditional convex body K in $\mathbb{R}^n$ we define a finite sequence $(K_j)_{j=1}^n$ of projections of the body K to the space spanned by first j vectors of the standard basis of $\mathbb{R}^n$. We prove that the sequence of volumes $(|K_j|)_{j=1}^n$ is log-concave.

1. Introduction

The main interest in convex geometry is the examination of sections and projections of sets. Some introduction can be found in a monograph by Gardner, [4]. We are interested in a class $\mathcal{PU}_n$ of convex bodies in $\mathbb{R}^n$ which are unconditional and permutationally invariant.

Let us briefly recall some definitions. A convex body K in $\mathbb{R}^n$ is called *unconditional* if for every point $(x_1, \ldots, x_n) \in K$ and every choice of signs $\epsilon_1, \ldots, \epsilon_n \in \{-1, 1\}$ the point $(\epsilon_1 x_1, \ldots, \epsilon_n x_n)$ also belongs to K. A convex body K in $\mathbb{R}^n$ is called *permutationally invariant* if for every point $(x_1, \ldots, x_n) \in K$ and every permutation $\pi : \{1, \ldots, n\} \longrightarrow \{1, \ldots, n\}$ the point $(x_{\pi(1)}, \ldots, x_{\pi(n)})$ is also in K. A sequence $(a_i)_{i=1}^n$ of positive real numbers is called *log-concave* if $a_i^2 \geq a_{i-1}a_{i+1}$, for $i = 2, \ldots, n-1$.

The main result of this paper reads as follows.

Theorem 1. *Let $n \geq 3$ and let $K \in \mathcal{PU}_n$. For each $i = 1, \ldots, n$ we define a convex body $K_i \in \mathcal{PU}_i$ as an orthogonal projection of K to the subspace $\{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_{i+1} = \ldots = x_n = 0\}$. Then the sequence of volumes $(|K_i|)_{i=1}^n$ is log-concave. In particular*

$$|K_{n-1}|^2 \geq |K_n| \cdot |K_{n-2}|. \tag{1}$$

Inequality (1) is related to the problem of negative correlation of coordinate functions on $K \in \mathcal{PU}_n$, i.e. the question whether for every $t_1, \ldots, t_n \geq 0$

$$\mu_K\left(\bigcap_{i=1}^n \{|x_i| \geq t_i\}\right) \leq \prod_{i=1}^n \mu_K\left(|x_i| \geq t_i\right), \tag{2}$$

where μ_K is normalized Lebesgue measure on K. Indeed, the Taylor expansion of the function $h(t) = \mu_K(|x_1| \geq t)\mu_K(|x_2| \geq t) - \mu_K(|x_1| \geq t, |x_2| \geq t)$ at $t = 0$ contains

$$\frac{1}{|K_n|^2}\left(|K_{n-1}|^2 - |K_{n-2}| \cdot |K_n|\right) t^2,$$

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cf. (1), as a leading term. The property (2), the so-called concentration hypothesis and the central limit theorem for convex bodies are closely related, see [1]. The last theorem has been recently proved by Klartag, [7].

The negative correlation property in the case of generalized Orlicz balls was originally investigated by Wojtaszczyk in [9]. A generalized Orlicz ball is a set

$$B = \left\{ (x_1, \ldots, x_n) \in \mathbb{R}^n \;\Big|\; \sum_{i=1}^{n} f_i(|x_i|) \leq n \right\},$$

where $f_1, \ldots, f_n$ are some Young functions (see [9] for the definition). In probabilistic terms Pilipczuk and Wojtaszczyk (see [8]) have shown that the random variable $X = (X_1, \ldots, X_n)$ uniformly distributed on B satisfies the inequality

$$\mathrm{Cov}(f(|X_{i_1}|, \ldots, |X_{i_k}|), g(|X_{j_1}|, \ldots, |X_{j_l}|)) \leq 0$$

for any bounded coordinate-wise increasing functions $f\colon \mathbb{R}^k \longrightarrow \mathbb{R}$, $g\colon \mathbb{R}^l \longrightarrow \mathbb{R}$ and any disjoint subsets $\{i_1, \ldots, i_k\}$ and $\{j_1, \ldots, j_l\}$ of $\{1, \ldots, n\}$. In the case of generalized isotropic Orlicz balls this result implies the inequality

$$\mathrm{Var}|X|^p \leq \frac{Cp^2}{n}\mathbb{E}|X|^{2p}, \qquad p \geq 2,$$

from which some reverse Hölder inequalities can be deduced (see [3]).

One may ask about an example of a nice class of Borel probability measures on $\mathbb{R}^n$ for which the negative correlation inequality hold. Considering the example of the measure with the density

$$p(x_1, \ldots, x_n) = \exp\left(-2(n!)^{1/n} \max\{|x_1|, \ldots, |x_n|\}\right),$$

which was mentioned by Bobkov and Nazarov in a different context (see [2, Lemma 3.1]), we certainly see that the class of unconditional and permutationally invariant log-concave measures would not be the answer. Nevertheless, it remains still open whether the negative correlation of coordinate functions holds for measures uniformly distributed on the bodies from the class $\mathcal{PU}_n$.

We should remark that our inequality (1) is similar to some auxiliary result by Giannopoulos, Hartzoulaki and Paouris, see [6, Lemma 4.1]. They proved that a version of inequality (1) holds, up to the multiplicative constant $\frac{n}{2(n-1)}$, for an arbitrary convex body.

The paper is organised as follows. In Section 2 we give the proof of Theorem 1. Section 3 is devoted to some remarks. Several examples are there provided as well.

2. Proof of the main result

Here we deal with the proof of Theorem 1. We start with an elementary lemma.

Lemma 1. *Let $f\colon [0, L] \longrightarrow [0, \infty)$ be a nonincreasing concave function such that $f(0) = 1$. Then*

$$\text{(3)} \qquad \frac{n-1}{n}\left(\int_0^L f(x)^{n-2}\mathrm{d}x\right)^2 \geq \int_0^L x f(x)^{n-2}\mathrm{d}x, \qquad n \geq 3.$$

Proof. By a linear change of a variable one can assume that $L = 1$. Since f is concave and nonincreasing, we have $1 - x \leq f(x) \leq x$ for $x \in [0, 1]$. Therefore, there exists a real number $\alpha \in [0, 1]$ such that for $g(x) = 1 - \alpha x$ we have

$$\int_0^1 f(x)^{n-2} \mathrm{d}x = \int_0^1 g(x)^{n-2} \mathrm{d}x.$$

Clearly, we can find a number $c \in [0, 1]$ such that $f(c) = g(c)$. Since f is concave and g is affine, we have $f(x) \geq g(x)$ for $x \in [0, c]$ and $f(x) \leq g(x)$ for $x \in [c, 1]$. Hence,

$$\begin{aligned}\int_0^1 x(f(x)^{n-2} - g(x)^{n-2})\mathrm{d}x \leq \int_0^c & c(f(x)^{n-2} - g(x)^{n-2})\mathrm{d}x \\ & + \int_c^1 c(f(x)^{n-2} - g(x)^{n-2})\mathrm{d}x \;=\; 0.\end{aligned}$$

We conclude that it suffices to prove (3) for the function g, which is by simple computation equivalent to

$$\begin{aligned}\frac{1}{\alpha^2 n(n-1)} \left(1 - (1-\alpha)^{n-1}\right)^2 \geq \frac{1}{\alpha^2} \Bigg(& \frac{1}{n-1} \left(1 - (1-\alpha)^{n-1}\right) \\ & - \frac{1}{n} \left(1 - (1-\alpha)^{n}\right) \Bigg).\end{aligned}$$

To finish the proof one has to perform a short calculation and use Bernoulli's inequality.

□

Proof of Theorem 1. Due to an inductive argument it is enough to prove inequality (1).

Let $g\colon \mathbb{R}^{n-1} \longrightarrow \{0, 1\}$ be a characteristic function of the set K_{n-1}. Then, by permutational invariance and unconditionality, we have

$$|K_{n-1}| = 2^{n-1}(n-1)! \int_{x_1 \geq \ldots \geq x_{n-1} \geq 0} g(x_1, \ldots, x_{n-1}) \mathrm{d}x_1 \ldots \mathrm{d}x_{n-1}, \tag{4}$$

and similarly

$$|K_{n-2}| = 2^{n-2}(n-2)! \int_{x_1 \geq \ldots \geq x_{n-2} \geq 0} g(x_1, \ldots, x_{n-2}, 0) \mathrm{d}x_1 \ldots \mathrm{d}x_{n-2}. \tag{5}$$

Moreover, permutational invariance and the definition of a projection imply

$$\mathbf{1}_{K_n}(x_1, \ldots, x_n) \leq \prod_{i=1}^{n} g(x_1, \ldots, \hat{x_i}, \ldots, x_n). \tag{6}$$

Thus

$$\begin{aligned}|K_n| &\leq 2^n n! \int_{x_1 \geq \ldots \geq x_n \geq 0} \prod_{i=1}^{n} g(x_1, \ldots, \hat{x_i}, \ldots, x_n) \mathrm{d}x_1 \ldots \mathrm{d}x_n \\ &= 2^n n! \int_{x_1 \geq \ldots \geq x_n \geq 0} g(x_1, \ldots, x_{n-1}) \mathrm{d}x_1 \ldots \mathrm{d}x_n \\ &= 2^n n! \int_{x_1 \geq \ldots \geq x_{n-1} \geq 0} x_{n-1} g(x_1, \ldots, x_{n-1}) \mathrm{d}x_1 \ldots \mathrm{d}x_{n-1},\end{aligned} \tag{7}$$

where the first equality follows from the monotonicity of the function g for nonnegative arguments with respect to each coordinate. We define a function $F\colon [0,\infty) \longrightarrow [0,\infty)$ by the equation

$$F(x) = \frac{\int_{x_1 \geq \ldots \geq x_{n-2} \geq x} g(x_1, \ldots, x_{n-2}, x)\mathrm{d}x_1 \ldots \mathrm{d}x_{n-2}}{\int_{x_1 \geq \ldots \geq x_{n-2} \geq 0} g(x_1, \ldots, x_{n-2}, 0)\mathrm{d}x_1 \ldots \mathrm{d}x_{n-2}}.$$

One can notice that

1. $F(0) = 1$.
2. The function F is nonincreasing as so is the function
$$x \mapsto g(x_1, \ldots, x_{n-2}, x)\mathbf{1}_{\{x_1 \geq \ldots \geq x_{n-2} \geq x\}}.$$
3. The function $F^{1/(n-2)}$ is concave on its support $[0, L]$ since $F(x)$ multiplied by some constant equals the volume of the intersection of the convex set $K_{n-1} \cap \{x_1 \geq \ldots \geq x_{n-1} \geq 0\}$ with the hyperplane $\{x_{n-1} = x\}$. This is a simple consequence of the Brunn-Minkowski inequality, see for instance [5, page 361].

By the definition of the function F and equations (4), (5) we obtain

$$\int_0^L F(x)\mathrm{d}x = \frac{\frac{1}{2^{n-1}(n-1)!}|K_{n-1}|}{\frac{1}{2^{n-2}(n-2)!}|K_{n-2}|} = \frac{1}{2(n-1)} \cdot \frac{|K_{n-1}|}{|K_{n-2}|},$$

and using inequality (7)

$$\int_0^L xF(x)\mathrm{d}x \geq \frac{\frac{1}{2^n n!}|K_n|}{\frac{1}{2^{n-2}(n-2)!}|K_{n-2}|} = \frac{1}{2^2 n(n-1)} \cdot \frac{|K_n|}{|K_{n-2}|}.$$

Therefore it is enough to show that

$$\frac{n-1}{n}\left(\int_0^L F(x)\mathrm{d}x\right)^2 \geq \int_0^L xF(x)\mathrm{d}x.$$

This inequality follows from Lemma 1. □

3. Some remarks

In this section we give some remarks concerning Theorem 1.

Remark 1. Apart from the trivial example of the B_∞^n ball, there are many other examples of bodies for which equality in (1) is attained. Indeed, analysing the proof, we observe that for the equality in (1) the equality in Lemma 1 is needed. Therefore, the function $F^{1/(n-2)}$ has to be linear and equal to $1 - x$. Taking into account the equality conditions in the Brunn-Minkowski inequality (consult [5, page 363]), this is the case if and only if the set $K_{n-1} \cap \{x_1 \geq \ldots \geq x_{n-1} \geq 0\}$ is a cone C with the base $(K_{n-2} \cap \{x_1 \geq \ldots \geq x_{n-2} \geq 0\}) \times \{0\} \subset \mathbb{R}^{n-1}$ and the vertex $(z_0, \ldots, z_0) \in \mathbb{R}^{n-1}$. Thus if for a convex body $K \in \mathcal{PU}_n$ we have the equality in (1), then this body K is constructed in the following manner. Take an arbitrary $K_{n-2} \in \mathcal{PU}_{n-2}$. Define the set K_{n-1} as the smallest permutationally invariant unconditional body containing C. For z_0 from some interval the set K_{n-1} is convex. For the characteristic function of the body K we then set $\prod_{i=1}^n \mathbf{1}_{K_{n-1}}(x_1, \ldots, \hat{x_i}, \ldots, x_n)$.

A one more natural question to ask is when a sequence $(|K_i|)_{i=1}^n$ is geometric Bearing in mind what has been said above for $i = 2, 3, \ldots, n-1$ we find that a sequence $(|K_i|)_{i=1}^n$ is geometric if and only if

$$K = [-L, L]^n \ \cup \bigcup_{i\in\{1,\ldots,n\},\epsilon\in\{-1,1\}} \operatorname{conv}\{\epsilon a e_i, \{x_i = \epsilon L, |x_k| \leq L, k \neq i\}\},$$

for some positive parameters a and L satisfying $L < a < 2L$, where $e_1, \ldots, e_n$ stand for the standard orthonormal basis in $\mathbb{R}^n$. One can easily check that $|K_i| = 2^i L^{i-1} a$.

Remark 2. Suppose we have a sequence of convex bodies $K_n \in \mathcal{PU}_n$, for $n \geq 1$, such that $K_n = \pi_n(K_{n+1})$, where by $\pi_n \colon \mathbb{R}^{n+1} \longrightarrow \mathbb{R}^n$ we denote the projection $\pi_n(x_1, \ldots, x_n, x_{n+1}) = (x_1, \ldots, x_n)$. Since Theorem 1 implies that the sequence $(|K_n|)_{n=1}^\infty$ is log-concave we deduce the existence of the limits

$$\lim_{n\to\infty} \frac{|K_{n+1}|}{|K_n|}, \qquad \lim_{n\to\infty} \sqrt[n]{|K_n|}.$$

We can obtain this kind of sequences as finite dimensional projections of an Orlicz ball in ℓ_∞.

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