Liga zadaniowa 2017

seria VI, opublikowane 16/05/2017

Pytania dotyczące zadań prosimy kierować do Piotra Nayara na adres: nayar@mimuw.edu.pl.

Problem 1. (**A**) For a function $g : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ we define

$$(\partial g)(x) = \{y \in \mathbb{R} : \ g(z) \geq g(x) + y(z - x), \quad \text{for any } z \in \mathbb{R}\}.$$

Let $A \subseteq \mathbb{R}^2$. Prove that the following conditions are equivalent,

(a) for any $m \geq 1$ and $(x_1, y_1), \ldots, (x_m, y_m) \in A$ we have

$$y_1(x_1 - x_2) + y_2(x_2 - x_3) + \ldots + y_m(x_m - x_1) \leq 0,$$

(b) there exists a function $g : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ such that g is not identically equal $+\infty$ and $A \subseteq \{(x, y) : \ y \in (\partial g)(x)\}$.

Problem 2. (**G**) Let A be a $n \times n$ real matrix with $\mathrm{tr}(A) = 0$.

(a) Show that there are real $n \times n$ matrices X, Y such that $A = XY - YX$.

(b) Show that there is an invertible real matrix C such that CAC^{-1} has zero diagonal.

Problem 3. (**A**) Let $a_1, \ldots, a_k$ be real numbers and let $n_1 < n_2 < \ldots < n_k$ be positive integers. Prove that

$$\int_{-\pi}^{\pi} \left| \sum_{i=1}^{k} a_i \sin n_i t \right| \mathrm{d}t \geq \pi \max_{1 \leq i \leq k} |a_i|.$$

Problem 4. (**K**) Let $G = (V, E)$ be a simple graph with a maximal vertex degree Δ and suppose we are give a set of colours $C = \{c_1, \ldots, c_q\}$, where $|C| = q = \Delta + 2$. A colouring $h : V \to C$ is *proper* if $x \sim y$ implies $h(x) \neq h(y)$. Let $\mathcal{H}$ be the set of all proper colourings of G. We introduce a graph $\mathcal{G} = (\mathcal{H}, \mathcal{E})$, where $h_1, h_2 \in \mathcal{H}$ are neighbours, that is $(h_1, h_2) \in \mathcal{E}$, if and only if $|\{v \in V : h_1(v) \neq h_2(v)\}| = 1$. Is $\mathcal{G}$ always connected?