Liga zadaniowa 2017

seria V, opublikowane 05/05/2017

Pytania dotyczące zadań prosimy kierować do Piotra Nayara na adres: nayar@mimuw.edu.pl.

Problem 1.

(a) Let $I_1, \ldots, I_k$ be closed intervals in $\mathbb{R}$ with lengths $l_1, \ldots, l_k$ and midpoints $x_1, \ldots, x_k$. Assume that $I = \bigcup_{i=1}^k I_i$ is a closed interval. Show that the closed interval centred at $x = \frac{\sum_{i=1}^k l_i x_i}{\sum_{i=1}^k l_i}$ with length $\sum_{i=1}^k l_i$ covers I.

(b) Let $B_1, \ldots, B_k$ be closed Euclidean balls in $\mathbb{R}^n$ with diameters $d_1, \ldots, d_k$. Assume no hyperplane divides the balls into two non-empty sets without intersecting at least one ball. Prove that the balls can be covered by a closed ball of diameter $\sum_{i=1}^k d_i$.

Problem 2. (**G**) Prove that there cannot be more than $\binom{n+1}{2}$ equiangular lines in $\mathbb{R}^n$. Is this result sharp for $n = 3$?

Problem 3. (**G**) Let A be a real symmetric $n \times n$ matrix, not equal to the zero matrix. Then

$$\operatorname{rank}(A) \geq \frac{\left(\sum_{i=1}^n a_{ii}\right)^2}{\sum_{i,j=1}^n a_{ij}^2}.$$

Problem 4. (**A+K**) A sequence of real numbers $(x_n)_{n=1}^\infty \subset [0,1]$ is called *nice* if for any two closed intervals $I, J \subset [0,1]$ of the same length we have

$$|\#\{1 \leq i \leq n : x_i \in I\} - \#\{1 \leq i \leq n : x_i \in J\}| \leq 10^6.$$

Prove that nice sequences do not exist.