Liga zadaniowa 2017

seria IV, opublikowane 10/04/2017

Pytania dotyczące zadań prosimy kierować do Piotra Nayara na adres: nayar@mimuw.edu.pl.

Problem 1. (**A+G**) Let P be a non-constant polynomial with complex coefficients. Show that the roots[1] of P' lie in the convex hull of the set of roots of P.

Problem 2. (**G+K**) Let n be a positive integer and let $G = (V, E)$ be a bipartite graph with bipartition $V = \{u_1, \ldots, u_n\} \cup \{v_1, \ldots, v_n\}$ and with m edges. For every edge $\{u_i, u_j\} \in E$ we introduce a variable x_{ij}. Define a matrix[2] $A = (a_{ij})_{i,j=1}^n$ by

$$a_{ij} = \begin{cases} x_{ij} & \text{when } \{u_i, u_j\} \in E \\ 0 & \text{otherwise} \end{cases} .$$

The determinant $\det(A)$ is a polynomial in m variables x_{ij}. Show that this polynomial is identically zero if and only if G has no perfect matching[3].

Problem 3. Suppose there is a "linear" parking lot with n available parking spots (these spots have numbers $1, \ldots, n$). There are n cars waiting in the line to park in this parking lot. Assume that the ith car prefers spot number $a_i \in \{1, \ldots, n\}$. If the spot is already occupied, then the ith car takes the next available space (having number greater than a_i) . If this is not possible, then it fails to park. What is the number of sequences $(a_i)_{i=1}^n$ such that all cars can park?

Example 1. Suppose that $a_1 = 1, a_2 = 1, a_3 = 2$. Then the first car takes spot number 1. The second car also wants to take this spot, but since it is not available, it takes the next available spot, which is spot number 2. In the same manner, the car number 3 takes spot number 3, since spot number 2 is occupied.

If $a_1 = 2, a_2 = 2, a_3 = 3$, then the first car takes spot number 2, the second car takes spot number 3 (since spot number 2 is occupied), and the third car fails to park.

Problem 4. (**A+G**) Let $(a_{ij})_{i,j=1}^n$ be a positive semidefinite matrix. Is it true that the matrix $(e^{a_{ij}})_{i,j=1}^n$ is also positive semidefinite?

[1] Here by *root* we mean any $z \in \mathbb{C}$ such that $P(z) = 0$.

[2] Note that $x_{ij} = x_{ji}$ and so A is symmetric.

[3] A *matching* in a graph G is a set of edges $F \subseteq E(G)$ such that no vertex of G is incident to more than one edge of F. A *perfect matching* is a matching covering all vertices.