Liga zadaniowa 2017

seria III, opublikowane 27/03/2017

Pytania dotyczące zadań prosimy kierować do Piotra Nayara na adres: nayar@mimuw.edu.pl.

Problem 1. (**K**) Let G be a simple triangle-free graph with n vertices. Suppose that any vertex of G has degree greater than $\frac{2n}{5}$. Show that G is bipartite.

Problem 2. (**G**) Suppose A and B are $n \times n$ real matrices with A positive semidefinite and B symmetric. Prove that the characteristic polynomial of AB is real rooted.

Problem 3. (**A**) Let E be the arithmetic mean and M be a median of a sequence of real numbers $a_1, \ldots, a_n$. Prove that for any $p \geq 1$ we have

$$\frac{1}{3^p} \sum_{k=1}^{n} |a_k - M|^p \leq \sum_{k=1}^{n} |a_k - E|^p \leq 2^p \sum_{k=1}^{n} |a_k - M|^p.$$

Problem 4. (**A+G+K**) Let $v_1, \ldots, v_n$ be unit vectors in $\mathbb{R}^n$, such that for all $i, j = 1, \ldots, n$ we have $\langle v_i, v_j \rangle \geq -0.999$. Prove that there is a vector $x \in \mathbb{R}^n$ such that

$$|\{1 \leq i \leq n : \ \langle x, v_i \rangle > 0\}| \geq \frac{1}{2}(n + n^{0.001}).$$