Liga zadaniowa 2017

seria II, opublikowane 13/03/2017

Pytania dotyczące zadań prosimy kierować do Piotra Nayara na adres: nayar@mimuw.edu.pl.

Problem 1. (**A+G**)

(a) Let A be a positive definite $n \times n$ matrix. Prove that

$$\ln \det A = \inf_{B \text{ positive definite}} \left(\operatorname{tr}(AB) - n - \ln \det B \right).$$

(b) Deduce that for any positive semi-definite $n \times n$ matrices A, B we have

$$\det(\lambda A + (1-\lambda)B) \geq \det(A)^{\lambda} \det(B)^{1-\lambda}, \qquad \lambda \in [0,1].$$

Problem 2. (**K+A**) Consider the following process. We start with an urn containing two balls, one black and one white. In each step of the process we choose a ball at random from those already in the urn and return the chosen ball to the urn together with additional ball of the same color. Let $n \geq 1$ and let $j \in \{1, \ldots, n+1\}$. What is the probability that after n steps of this process the number of black balls inside the urn is equal to j?

Problem 3. (**G**) Fix a sequence of pairwise distinct positive real numbers $x_1, \ldots, x_n$. Prove that the sign of the determinant of the matrix

$$\begin{pmatrix} x_1^{k_1} & x_1^{k_2} & \ldots & x_1^{k_n} \\ x_2^{k_1} & x_2^{k_2} & \ldots & x_2^{k_n} \\ \vdots & \vdots & \ddots & \vdots \\ x_n^{k_1} & x_n^{k_2} & \ldots & x_n^{k_n} \end{pmatrix}$$

does not depend on the choice of integers $0 \leq k_1 < k_2 < \ldots < k_n$.

Problem 4. (**G+K**) Let $G = (V, E)$ be a simple connected and undirected graph with $|V| = n$. For $1 \leq i, j \leq n$ let d_{ij} be the distance between vertices v_i, v_j in the standard graph metric. We define the matrix $D(G) = (d_{ij})_{i,j=1}^{n}$. Show that for any tree T we have $\det(D(T)) = (-1)^{n-1}(n-1)2^{n-2}$.