Liga zadaniowa 2017

seria I, opublikowane 27/02/2017

Pytania dotyczące zadań prosimy kierować do Piotra Nayara na adres: nayar@mimuw.edu.pl.

Problem 1. (**A+G**) Let $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ be positive real numbers such that $\sum_{i=1}^n a_i > \sum_{i=1}^n b_i$. Let us take the boxes $A = [0, a_1] \times \ldots \times [0, a_n]$ and $B = [0, b_1] \times \ldots \times [0, b_n]$. Prove that for any isometry R of $\mathbb{R}^n$ we have $R(A) \nsubseteq B$. In other words, the box A cannot be packed in the box B.

Problem 2. (**G**) Show that the rectangle $[0, 1] \times [0, a]$ can be expressed as a finite sum of closed squares (having sides parallel to the coordinate axes) with pairwise disjoin interiors if and only if a is rational.

Problem 3. (**A+K**) Let $f : \mathbb{Z}^2 \to \mathbb{R}$ be such that $f(N, 0) = 1$ and $f(-N, 0) = -1$. Show that

$$\sum_{x,y\in\mathbb{Z}^2:\ |x-y|=1} (f(x) - f(y))^2 \geq \frac{c}{\ln N},$$

where $|\cdot|$ is the standard Euclidean norm and c is a universal constant (for example $c = 10^{-10}$ should work).

Problem 4. (**K**) Fix positive integers n and m with $n < m$. Show that among all up-right $\mathbb{Z}^2$ lattice paths from $(0, 0)$ to (n, m) the fraction of paths lying strictly above the line $x = y$ (except for the initial point) is equal to $\frac{m-n}{m+n}$.