Inequalities in convex geometry
homework problems

Problem 1. (3 p.) Let K be a convex body in $\mathbb{R}^n$ with volume 1 and barycentre at the origin, that is $\int_K x_i \mathrm{d}x = 0$ for all $i = 1, \ldots, n$. Prove that for any $\theta \in S^{n-1}$ we have

$$\mathrm{vol}_n(\{x \in K : \langle x, \theta \rangle \geq 0\}) \geq \left(\frac{n}{n+1}\right)^n.$$

Hint: use Brunn's principle.

Problem 2. (3 p.)

(a) Prove that Steiner symmetrization of a compact convex set is compact.
(b) Is it true that Steiner symmetrization preserves compactness?

Problem 3. (3 p.) Prove Blaschke selection principle using Arzela-Ascoli theorem.

Problem 4. (1 p.) Prove that if K, L are compact sets such that $|K| = |L|$, then for any $t \geq 0$ we have $|K + tL| \geq |(1+t)L|$.

Problem 5. (2 p.) Prove that for compact convex bodies with non-empty interiors the convergence $K_j \to K$ implies $S_u(K_j) \to S_u(K)$ for any direction $u \in S^{n-1}$.

Problem 6. (3 p.)

(a) Suppose K, L are compact sets in $\mathbb{R}^n$. Prove the inequality

$$\mathrm{vol}_n(K)\,\mathrm{vol}_n(L) \leq \mathrm{vol}_n(K+L) \sup_{x \in \mathbb{R}^n} \mathrm{vol}_n(K \cap (x - L)).$$

(b) Show that if K is convex then

$$\sup_{x \in \mathbb{R}^n} \mathrm{vol}_n(K \cap (x - K)) \geq 2^{-n}\,\mathrm{vol}_n(K).$$

(c) Prove that every convex body K contains a symmetric (with respect to some point) convex body L with

$$\mathrm{vol}_n(L) \geq 2^{-n}\,\mathrm{vol}_n(K).$$

Problem 7. (2 p.) Let K, L be convex bodies in $\mathbb{R}^n$. Prove the inequality

$$\mathrm{vol}_n(K - L) \leq \binom{2n}{n} \mathrm{vol}_n(K + L).$$

Problem 8. (2 p.) Suppose $f : \mathbb{R} \to \mathbb{R}$ be an even log-concave probability density. Prove or disprove the inequality

$$\int_t^{+\infty} f(s)\mathrm{d}s \leq \frac{f(t)}{2f(0)}.$$

Problem 9. (4 p.) Prove that in dimension one Prékopa-Leindler inequality for smooth strictly positive functions implies Prékopa-Leindler inequality for general measurable functions.

Problem 10. (1 p.) Let $(a_{ij})_{i,j=1}^n$ be a positive semi definite matrix. Prove that $(e^{a_{ij}})_{i,j=1}^n$ is also positive semi-definite.

Problem 11. (2 p.) Let A, B be symmetric positive semi-definite matrices. Prove that
$$\det(A+B)^{1/n} \geq \det(A)^{1/n} + \det(B)^{1/n}.$$

Problem 12. Let f be a bounded measurable function. Define
$$(Q_t f)(x) = \int_{\mathbb{R}^n} f(x + \sqrt{t}z)\mathrm{d}\gamma_n(z), \qquad t \geq 0$$

(i) (1 p.) Prove that for $t > 0$ the function $Q_t f$ is C^∞ smooth.
(ii) (1 p.) Prove that $Q_{s+t} = Q_s \circ Q_t$.
(iii) (1 p.) Prove that $\int f Q_t g = \int g Q_t f$ for $t \geq 0$.
(iv) (1 p.) Prove that $(Q_t f)(x) \sim_{t\to\infty} \frac{1}{(2\pi t)^{n/2}} \int f(x)\mathrm{d}x$.

Problem 13. (2 p.) Let X be a topological space, and M a family of closed subsets of X with the finite intersection property, i.e. with the property that every finite intersection of elements of M is non-empty. Is it true that X is compact if and only if every family of closed sets with the finite intersection property have non-empty intersection?

Problem 14. (3 p.) Suppose X is a compact topological space and let $g : X \to \mathbb{R}$ be an upper semi-continuous functional (that is, the sets of the form $\{g \geq t\}$ are closed).

(i) Is g bounded from above on X?
(ii) Assuming that g is bounded from above, does g attain its supremum on X?

Problem 15. (5 p.) Let M be a metric space and let $f : M \to \mathbb{R}$ be an upper semi-continuous function. Prove that there exists a sequence of continuous function $f_n : M \to \mathbb{R}$ such that $f_n \searrow f$.

Problem 16. (3 p.) Suppose μ is a measure with continuous density f. Assume μ is a log-concave measure on $\mathbb{R}^n$, that is
$$\mu(\lambda A + (1-\lambda)B) \geq \mu(A)^\lambda \mu(B)^{1-\lambda}, \qquad \lambda \in [0,1]$$
for any compact sets A, B in $\mathbb{R}^n$. Prove that $f = e^{-V}$ for some convex function $V : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$.

Problem 17. (1 p.) Find the John ellipsoid for the set $B_1^n = \{|x_1| + \ldots + |x_n| \leq 1\}$.

Problem 18. (2 p.) Suppose an ellipsoid is invariant under coordinate reflections and permuting coordinates. Does it follow that this ellipsoid is a centered Euclidean ball?

Problem 19. For a random vector X with density f on Euclidean space we write $h(X) = -\int f \ln f$, provided that this integral exists.

(i) (1 p.) For a random vector $X = (X_1, X_2)$, where $X_i \in R^{n_i}$, $i = 1, 2$ prove that

$$h(X_1, X_2) \leq h(X_1) + h(X_2).$$

(ii) (2 p.) For a random vector $X = (X_1, X_2, X_3)$, where $X_i \in R^{n_i}$, $i = 1, 2, 3$ prove that

$$h(X_1, X_2, X_3) + h(X_3) \leq h(X_1, X_3) + h(X_2, X_3).$$

(iii) (3 p.) For independent random vectors $X_i \in R^n$, $i = 1, 2, 3$ prove that

$$h(X_1 + X_2 + X_3) + h(X_3) \leq h(X_1 + X_3) + h(X_2 + X_3).$$

Problem 20. (4 p.) Suppose X is a random vector in $\mathbb{R}^n$ with density of the form

$$f(x) = \exp\left(-V(x) - \frac{1}{2}|x|^2\right)$$

for some convex function V. Suppose $\mathbb{E}X = 0$. Prove that for every convex function $\varphi : \mathbb{R}^n \to \mathbb{R}$ we have

$$\mathbb{E}\varphi(X) \leq \mathbb{E}\varphi(G),$$

where $G \sim \mathcal{N}(0, I_n)$ is a standard Gaussian random vector in $\mathbb{R}^n$.

Problem 21. Let us define Hermite polynomials via

$$H_n(x) = (-1)^n e^{x^2/2} \frac{\mathrm{d}^n}{\mathrm{d}x^n}(e^{-x^2/2}).$$

(i) (1 p.) Prove that $H_n(x) = \left(x - \frac{\mathrm{d}}{\mathrm{d}x}\right)^n \cdot 1$.

(ii) (1 p.) Prove that $(H_n/\sqrt{n!})$ form an orthonormal system in $L_2\left(\mathbb{R}, \frac{1}{\sqrt{2\pi}}e^{-x^2/2}\right)$.

(iii) (1 p.) Prove that $H_n'(x) = nH_{n-1}(x)$.

Problem 22. (3 p.) Suppose G is a standard $\mathcal{N}(0,1)$ Gaussian random variable. Prove that for a compactly supported C^∞ function $f : \mathbb{R} \to \mathbb{R}$ we have

$$\operatorname{Var} f(G) = \mathbb{E}(f'(G))^2 - \sum_{k=2}^{\infty} \frac{k-1}{k!}(\mathbb{E}f^{(k)}(G))^2.$$

Problem 23. (4 p.) Let us define

$$M_\theta^p(x, y) = \begin{cases} (\theta x^p + (1-\theta)y^p)^{1/p} & p \in \mathbb{R} \setminus \{0\} \\ x^\theta y^{1-\theta} & p = 0 \\ \min\{x, y\} & p = -\infty \\ \max\{x, y\} & p = +\infty \end{cases},$$

for $x, y > 0$. If $x \cdot y = 0$, we set $M_\theta^p(x, y) = 0$.

Let $\gamma \geq -1/n$ and let $f, g, m : \mathbb{R}^n \to \mathbb{R}_+$ be positive measurable functions. Fix $\theta \in [0, 1]$. Assume that for all $x, y \in \mathbb{R}^n$ we have

$$m(\theta x + (1-\theta)y) \geq M_\theta^\gamma(f(x), g(y)).$$

Prove that

$$\int_{\mathbb{R}^n} m(x)\,\mathrm{d}x \geq M_\theta^{\frac{\gamma}{1+\gamma n}}\left(\int_{\mathbb{R}^n} f(x)\,\mathrm{d}x, \int_{\mathbb{R}^n} g(x)\,\mathrm{d}x\right).$$

Problem 24. (3 p.) Let A be a measurable set in $\mathbb{R}^n$ and let $r \geq 0$ be such that $\gamma_n(A) = \gamma_n(rB_2^n)$. Here γ_n stands for the standard Gaussian measure in $\mathbb{R}^n$. Prove that

$$\gamma_n(tA) \leq \gamma_n(t \cdot rB_2^n), \qquad t \geq 1.$$

Problem 25. (3 p.) Let f be a log-concave probability density on $\mathbb{R}$. Let $h(f) = -\int f \ln f$. Prove that

$$-\ln \|f\|_\infty \leq h(f) \leq 1 - \ln \|f\|_\infty .$$

Problem 26.

(i) (1 p.) Suppose $v_1, \ldots, v_n$ are vectors in $\mathbb{R}^n$. Prove that these vectors form an orthonormal basis if and only if $\sum_{i=1}^n v_i \otimes v_i = I_n$.

(ii) (1 p.) Suppose that $v_1, \ldots, v_m$ are vectors in $\mathbb{R}^n$ such that $\sum_{i=1}^m v_i \otimes v_i = I_n$. Prove that $m \geq n$.

(iii) (2 p.) Let $m \geq n$. Suppose $u_1, \ldots, u_m$ be an orthonormal basis in $\mathbb{R}^m$. Let $P : \mathbb{R}^m \to \mathbb{R}^n$ be given by $P(x_1, \ldots, x_m) = (x_1, \ldots, x_n)$. Prove that $\sum_{i=1}^m (Pu_i) \otimes (Pu_i) = I_n$.

(iv) (2 p.) Suppose $m \geq n$. Assume that $v_1, \ldots, v_m$ are vectors in $\mathbb{R}^n$ satisfying $\sum_{i=1}^m v_i \otimes v_i = I_n$. Does there always exist an orthonormal basis $u_1, \ldots, u_m$ of $\mathbb{R}^m$ such that for the standard projection P defined above we have $v_i = Pu_i$ for $i = 1, \ldots, m$?

Problem 27. (3 p.) Suppose X, Y are positive i.i.d. real random variables having densities with respect to the Lebesgue measures. Find the best constant C is the inequality

$$H\left(\frac{X}{X+Y}\right) \leq C.$$

Problem 28. (2 p.) Let X be a real random variable and let $\psi : \mathbb{R} \to \mathbb{R}$ be a C^1 increasing map. Prove that

$$H(\psi(X)) = H(X) + \mathbb{E} \ln \psi'(X).$$

In particular

$$H(e^X) = H(X) + \mathbb{E}X.$$

Problem 29. (4 p.) Take $n \geq 2$ and consider projections $\pi_j : \mathbb{R}^n \to \mathbb{R}^{n-1}$ given by

$$\pi_j(x_1, \ldots, x_n) = (x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n).$$

Using Brascamp-Lieb inequality prove that for any functions $f_1, \ldots, f_n$ in $L^{n-1}(\mathbb{R}^{n-1})$ we have

$$\int_{\mathbb{R}^n} \prod_{j=1}^n f_j(\pi_j(x))\mathrm{d}x \leq \prod_{j=1}^n \|f_j\|_{L^{d-1}(\mathbb{R}^{d-1})} .$$

Deduce that for any measurable set A in $\mathbb{R}^n$ we have

$$\mathrm{vol}_n(A)^{n-1} \leq \prod_{j=1}^n \mathrm{vol}_{n-1}(\pi_j(A)).$$

Problem 30. (3 p.) Let $A_1, \ldots, A_n$ be symmetric positive definite $k \times k$ matrices. Prove that the function

$$(t_1, \ldots, t_n) \mapsto \ln \det \left(\sum_{i=1}^{n} e^{t_i} A_i \right)$$

is convex on $\mathbb{R}^n$.

Problem 31.

(i) (4 p.) Suppose $f : [0, \infty) \to [0, \infty)$ is a continuous log-concave function. Prove that the function

$$f(p) = \frac{1}{\Gamma(p+1)} \int_0^\infty t^p f(t) \mathrm{d}t, \qquad p > -1$$

is log-concave.

(ii) (1 p.) Prove that $\lim_{p \to -1} f(p) = f(0)$.

(iii) (1 p.) Prove that if $f(0) = 1$ then

$$p \mapsto \left(\frac{\int_0^\infty t^p f(t) \mathrm{d}t}{\Gamma(p+1)} \right)^{1/(p+1)}$$

is non-increasing on $(-1, \infty)$.

Problem 32. (2 p.) Suppose Z is a positive real random variable with log-concave tail, i.e., the function $f(t) = \mathbb{P}(Z > t)$ is log-concave. Let $\mathcal{E}$ be the exponential random variable with parameter 1. Prove that

$$(\mathbb{E} Z^p)^{1/p} \leq \frac{(\mathbb{E} \mathcal{E}^p)^{1/p}}{(\mathbb{E} \mathcal{E}^q)^{1/q}} (\mathbb{E} Z^q)^{1/q}, \qquad p \geq q > 0.$$

Problem 33. (3 p.) Let $h : \mathbb{R}_+ \to \mathbb{R}_+$ be a non-increasing function. Suppose $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$ be such that $\Phi(0) = 0$ and the function $\Phi(x)/x$ is non-decreasing. Prove that the function

$$p \mapsto \left(\frac{\int_0^\infty h(\Phi(x)) x^p \mathrm{d}x}{\int_0^\infty h(x) x^p \mathrm{d}x} \right)^{1/(p+1)}$$

is non-increasing on $(-1, \infty)$.

Problem 34. (3 p.) Let ϕ be a non-negative concave function supported on a convex body $K = \{\phi > 0\}$ in $\mathbb{R}^n$. Then for any $0 < p \leq q$ we have

$$\left(\binom{n+p}{n} \frac{1}{|K|} \int_K \phi(x)^p \mathrm{d}x \right)^{1/p} \geq \left(\binom{n+q}{n} \frac{1}{|K|} \int_K \phi(x)^q \mathrm{d}x \right)^{1/q}.$$

Problem 35. (2 p.) Let ν be a measure with density $\frac{1}{2} e^{-|x|}$ on $\mathbb{R}$. Prove that for any C^1 function f with compact support we have

$$\mathrm{Var}_\nu(f) \leq 4 \int (f')^2 \mathrm{d}\nu.$$

Problem 36. (2 p.) Suppose X is a random vector in $\mathbb{R}^n$ with density f. We define $H(X) = -\int f \ln f$ whenever this quantity exists. Take $N(X) = \frac{1}{2\pi e}\exp(2H(X)/n)$. Let $\operatorname{Cov}(X)$ be the covariance matrix of X. Prove that

$$N(X) \leq \det(\operatorname{Cov}(X))^{1/n}.$$