Exercise 1. Suppose X, Y are independent and identically distributed discrete random variables. Show that $\mathbb{P}(X = Y) \geq e^{-H(X)}$.

Exercise 2. Show that for any discrete random variable X (having values in a finite space M) and for any function $f : M \to M'$ we have $H(f(X)) \leq H(X)$.

Exercise 3. Show that for any discrete random variables $X_1, \ldots, X_n : \Omega \to \mathbb{R}$ we have $H(X_1 + \ldots + X_n) \leq H(X_1) + \ldots + H(X_n)$.

Exercise 4. Take $q > 1$. For a discrete random variable X let us define the q-entropy,

$$H_q(X) = \frac{1 - \sum_{i=1}^n p_i^q}{q-1}.$$

(a) Observe that $\lim_{q\to 1} H_q(X) = H(X)$.
(b) Prove that $H_q(X, Y) \leq H_q(X) + H_q(Y)$ for any random variables X, Y.
(c) Show that $H_q(X, Y, Z) + H_q(Z) \leq H_q(X, Z) + H_q(Z, Y)$.

Exercise 5. For a discrete random variables $X_1, \ldots, X_n$ and a subset $S \subseteq [n]$ take $X_S = \{X_i : \ i \in S\}$. Define

$$h_k^{(n)} = \frac{1}{\binom{n}{k}} \sum_{S:|S|=k} \frac{H(X_S)}{k}.$$

Show that $h_1^{(n)} \geq h_2^{(n)} \geq \ldots \geq h_n^{(n)}$.

Exercise 6. Let K_3 be a triangle (complete graph with 3 vertices). Show that for any $L \geq 3$ there is a graph G with at most L edges such that $\mathrm{Hom}(K_3, G) \geq cL^{3/2}$, where c is a universal constant.

Exercise 7. Let $(p_i)_{i=1}^n$, $(q_i)_{i=1}^n$ be two sequences of nonnegative real numbers such that $\sum_{i=1}^n p_i = \sum_{i=1}^n q_i = 1$. Prove the inequality

$$\sum_{i=1}^n p_i \ln\left(\frac{p_i}{q_i}\right) \geq \frac{1}{2}\left(\sum_{i=1}^n |p_i - q_i|\right)^2.$$

Exercise 8. Let (X, μ) be a measure space and let $1 < p \leq 2$. Show that for $f, g \in L_p(X, \mu)$ we have

$$\|f+g\|_p^p + \|f-g\|_p^p \geq (\|f\|_p + \|g\|_p)^p + |\,\|f\|_p - \|g\|_p\,|^p.$$

Exercise 9.

(a) Let $A \subseteq \{-1, 1\}^n$ and let $v \in A$. Take $d_A(v) = |\{u \in A : \ u \sim v\}|$. Then

$$|A| \geq 2^{\bar{d}}, \qquad \text{where } \ \bar{d} = \frac{1}{|A|}\sum_{v\in A} d_A(v).$$

(b) Let $A \subseteq \{-1, 1\}^n$, $|A| = m$. Prove that $|E(A, A^c)| \geq m(n - \log_2 m)$.

(c) Deduce the inequality

$$E(A, A^c) \geq 2^n \mu_n(A) \log_2 \left(\frac{1}{\mu_n(A)} \right),$$

where μ_n is the uniform measure on the hypercube.

Exercise 10. We say that $f : \{-1,1\}^n \to \mathbb{R}$ is monotone if for any $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$ with $x_i \leq y_i$, $i = 1, \ldots, n$ we have $f(x) \leq f(y)$. Consider the Walsh-Fourier expansion of f, namely $f = \sum_S a_S w_S$.

(a) Prove that for any Boolean function $f : \{-1,1\}^n \to \{-1,1\}$ we have $I_i(f) \geq |a_{\{i\}}|$, $i = 1, \ldots, n$.
(b) Show that $I(f) \leq \deg(f)$, where $\deg(f)$ is the degree of the multivariate polynomial $\sum_S a_S w_S$.
(c) Prove that for monotone Boolean f we have $I_i(f) = a_{\{i\}}$.

Exercise 11. Let n be odd. Take $f : \{-1,1\}^n \to \{-1,1\}$ given by $f(x) = \mathrm{sgn}(x_1 + \ldots + x_n)$.

(a) Compute the influences of f.
(b) Show that for any monotone function $g : \{-1,1\}^n \to \{-1,1\}$ we have $I(g) \leq I(f)$.

Exercise 12. Let $f : \{-1,1\}^n \to \{-1,0,1\}$. Define $\mathrm{supp}(f) = \{x : f(x) \neq 0\}$.

(a) Show that for non zero f and any $\delta \in [0,1]$ we have $\delta^{\deg(f)} \leq \mu_n(\mathrm{supp}(f))^{\frac{1-\delta}{1+\delta}}$.
(b) Deduce that for non zero f we have $\mu_n(\mathrm{supp}(f)) \geq e^{-2\deg(f)}$.

Exercise 13. Suppose $f : \{-1,1\}^n \to \{0,1\}$ has the Walsh-Fourier expansion $f = \sum_S a_S w_S$. Show that $\sum_{i=1}^n a_{\{i\}}^2 \leq 2(\mathbb{E}f)^2 \ln(1/\mathbb{E}f)$.

Exercise 14. Let n be odd. Show that in the 3-candidate Condorcet elections using $f(x) = \mathrm{sgn}(x_1 + \ldots + x_n)$, the probability of a Condorcet winner tends to $\frac{3}{2\pi} \arccos(-1/3) \approx 91.2\%$, as $n \to \infty$.

Exercise 15. Take $n = mk$ and divide n variables into m groups (tribes), each of cardinality k. Define

$$\mathrm{Tribes}_{k,m}(x_1, \ldots, x_n) = \mathrm{OR}\left(\mathrm{AND}(x_1, ..., x_k), ..., \mathrm{AND}(x_{(m-1)k+1}, ..., x_{mk})\right).$$

Here $\mathrm{AND} : \{-1,1\}^k \to \{-1,1\}$, $\mathrm{AND}(x_1, ..., x_k) = \min\{x_1, \ldots, x_k\}$ and $\mathrm{OR} : \{-1,1\}^m \to \{-1,1\}$, $\mathrm{OR}(y_1, \ldots, y_m) = \max\{y_1, \ldots, y_m\}$.

(a) Compute $I_i(\mathrm{Tribes}_{k,m})$ and $I(\mathrm{Tribes}_{k,m})$.
(b) Compute $\mathbb{E}[\mathrm{Tribes}_{k,m}]$.
(c) Prove that for any $p \in [0,1]$ there is a sequence of functions $(f_n)_{n \geq 1}$, $f_n : \{-1,1\}^n \to \{-1,1\}$ such that $\lim_{n\to\infty} \mu_n(\{x : f_n(x) = 1\}) = p$ and

$$\max_{i=1,\ldots,n} I_i(f_n) \geq c \, \mathrm{Var}_{\mu_n}(f_n) \frac{\ln n}{n}.$$

Exercise 16. Let $f : \mathbb{R}^n \to [0,\infty)$ be compactly supported (to avoid any non-essential problems with integrability) with $\int_{\mathbb{R}^n} f = 1$. Define $S(f) = -\int_{\mathbb{R}^n} f \ln f$ and $N(f) = \exp(\frac{2}{n}S(f))$. Let $(P_t)_{g\geq 0}$ be the heat semigroup. Prove that the function $t \mapsto N(P_t(f))$ is concave.

Exercise 17. Assume that (X,Y) is a symmetric log-concave random vector in $\mathbb{R}^2$, such that $X \sim Y$. Prove that $S(X+Y) \leq S(2X)$.

Exercise 18. Let $f : \mathbb{R} \to [0,\infty)$ be such that $\int_{\mathbb{R}} f = 1$ and $f = e^{-V}$ for some convex function V. Let $\|f\|_\infty = \sup(f)$. Prove that

$$e^{-1}\|f\|_\infty \leq e^{-S(f)} \leq \|f\|_\infty.$$

Exercise 19. Let $A \subseteq \mathbb{R}^n$ and let $f : A \to [0,\infty)$ satisfy $\int_A f = 1$. Prove that $S(f) \leq \ln|A|$, where $|A|$ is the Lebesgue measure of A.

Exercise 20. Let (Ω,μ) be a probability space. Show that for any $f,g : \Omega \to [0,\infty)$ and any $\lambda \in [0,1]$ we have

$$\mathrm{Ent}_\mu(\lambda f + (1-\lambda)g) \leq \lambda\,\mathrm{Ent}_\mu(f) + (1-\lambda)\,\mathrm{Ent}_\mu(g),$$

where $\mathrm{Ent}_\mu(f) = \int f \ln f \mathrm{d}\mu - \left(\int f \mathrm{d}\mu\right)\ln\left(\int f\mathrm{d}\mu\right)$.

Exercise 21. We say that $f : \{-1,1\}^n \to \mathbb{R}$ is monotone if for any $x = (x_1,\ldots,x_n)$, $y = (y_1,\ldots,y_n)$ with $x_i \leq y_i$, $i = 1,\ldots,n$ we have $f(x) \leq f(y)$. Let P_t be the semigroup generated by the continuous time random walk on $\{-1,1\}^n$ and let μ_n be the uniform measure on $\{-1,1\}^n$.

(a) Prove that $t \mapsto \int P_t(f)P_t(g)\mathrm{d}\mu_n$ is non-increasing.
(b) Deduce that $\int fg\mathrm{d}\mu_n \geq \int f\mathrm{d}\mu_n \int g\mathrm{d}\mu_n$.

Exercise 22. Let $(P_t)_{t\geq 0}$ be the semigroup generated by some finite Markov chain with a spectral gap α and stationary measure π. Prove that

$$\mathrm{Ent}_\pi(P_t f) \leq e^{-2\alpha t}\,\mathrm{Ent}_\pi(f).$$

Exercise 23. Consider the symmetric random walk on the cyclic group $\{0,1,\ldots,n-1\}$, i.e., $p(n,n+1) = p(n,n-1) = 1/2$.

(a) Find the spectral gap of this chain.
(b) Show that the above chain mixes in time $O(n^2)$.

Exercise 24. Let $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$ be the eigenvalues of $I - \frac{1}{d}A$, where A is the adjacency matrix of a graph G.

(a) Prove that $\lambda_2 \leq \frac{n}{n-1}$ with equality for complete graph on n vertices.
(b) Suppose G is not a complete graph. Show that $\lambda_2 \leq 1$.