Selected theorems in mathematics

Part I, prepared by: Piotr Nayar

Problem 1. (15 points)

(a) Let $\mathbb{F}$ be an arbitrary field and let $P(x_1, \ldots, x_n)$ be a polynomial in $\mathbb{F}[x_1, \ldots, x_n]$. Suppose that the degree of P is $\sum_{i=1}^n k_i$, where each k_i is a non-negative integer and suppose that the coefficient of $x_1^{k_1} \cdot \ldots \cdot x_n^{k_n}$ is non-zero. Then for any subsets $A_1, \ldots, A_n$ of $\mathbb{F}$ satisfying $|A_i| \geq k_i + 1$ for all $i = 1, \ldots, n$, there exist $a_1 \in A_1, \ldots, a_n \in A_n$ such that $P(a_1, \ldots, a_n) \neq 0$.

(b) Suppose that the hyperplanes $H_1, \ldots, H_m \subset \mathbb{R}^n$ cover the set $\{0,1\}^n \backslash \{0\}$ and that $0 \notin \bigcup_{i=1}^m H_i$. Prove that $m \geq n$.

Problem 2. (15 points)

(a) Let f be a trigonometric polynomial of order n, i.e.,

$$f(x) = \sum_{k=0}^{n} (a_k \cos kx + b_k \sin kx).$$

Let us define the function $D_n(x) = \frac{1}{2} + \sum_{k=1}^{n-1} \cos kx + \frac{1}{2} \cos nx$ and the set $A_n = \{\frac{2k-1}{2n}\pi,\ k = 1, 2, \ldots, 2n\}$. Prove the identity

$$f(x) = a_n \cos nx + \frac{1}{n} \sum_{t \in A_n} f(t) D_n(x - t).$$

(b) Prove that $D_n(x) = \frac{\sin nx}{2\mathrm{tg}(x/2)}$. Prove the identities

$$f'(0) = \frac{1}{n} \sum_{t \in A_n} f(t) \frac{(-1)^{k+1}}{(2\sin(t/2))^2}, \qquad \sum_{t \in A_n} \frac{1}{(2\sin(t/2))^2} = n^2$$

and deduce that

$$f'(x) = \frac{1}{n} \sum_{t \in A_n} f(x+t) \frac{(-1)^{k+1}}{(2\sin(t/2))^2}.$$

(c) Show that for every non-decreasing convex function $\phi : [0, \infty) \to \mathbb{R}$ we have

$$\int_0^{2\pi} \phi\left(\left|\frac{f'(x)}{n}\right|\right) \mathrm{d}x \leq \int_0^{2\pi} \phi\left(|f(x)|\right) \mathrm{d}x.$$

Deduce that for $1 \leq p < \infty$ we have

$$\left(\int_0^{2\pi} |f'(x)|^p \, \mathrm{d}x\right)^{1/p} \leq n \left(\int_0^{2\pi} |f(x)|^p \, \mathrm{d}x\right)^{1/p}.$$

Moreover,

$$\max_{x\in[0,2\pi]} |f'(x)| \leq n \max_{x\in[0,2\pi]} |f(x)|.$$

Problem 3. (10 points)

(a) Let $\mathcal{A}$ be a family of subsets of $\{1, 2, \ldots, n\}$ such that for any pair of subsets $A, B \in \mathcal{A}$ we have $A \not\subseteq B$. Prove that

$$|\mathcal{A}| \leq \binom{n}{[n/2]}$$

and determine the extremal case.

(b) Let $v_1, \ldots, v_n$ be real numbers such that $|v_i| \geq 1$ for $i = 1, \ldots, n$. Define

$$A = \{x = (x_1, \ldots, x_n) \in \{-1, 1\}^n, \quad |v_1 x_1 + \cdots + v_n x_n| < 1\}.$$

Prove that $|A| \leq \binom{n}{[n/2]}$.

Selected theorems in mathematics

Part II, prepared by: Piotr Nayar

Problem 1. (5 points) Let $A = (a_{ij})_{i,j=1}^n$ be a symmetric matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$. Prove that $\sum_{i,j=1}^n a_{ij}^2 = \sum_{i=1}^n \lambda_i^2$.

Problem 2. (10 points) Take $d \geq 1$ and let us consider $x_1, x_2, \ldots, x_n \in \mathbb{R}^d$, where $n \geq d+2$. Prove that there exists a partition of $\{1, \ldots, n\}$ into two sets I, J such that the convex hulls of the sets $\{x_i : \ i \in I\}$ and $\{x_j : \ j \in J\}$ have a nonempty intersection.

Problem 3. (10 points) Let $d \geq 1$ and let $A \subset \mathbb{R}^d$. Suppose $x \in \mathrm{conv}(A)$. Prove that there exists a set $B \subset A$ with $\#B \leq d+1$ such that $x \in \mathrm{conv}(B)$.

Problem 4. (10 points) Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function with period 1 and let α be irrational. Prove that

$$\lim_{n \to \infty} \frac{f(\alpha) + f(2\alpha) + \ldots + f(n\alpha)}{n} = \int_0^1 f(t) \, \mathrm{d}t.$$

Prove that for every interval $[a,b] \subset [0,1]$ and every irrational real number α we have

$$\lim_{n \to \infty} \frac{\#\{1 \leq k \leq n : \ \{k\alpha\} \in [a,b]\}}{n} = b - a,$$

where $\{x\} \in [0,1)$ is the fractional part of $x \in \mathbb{R}$.

Problem 5. (10 points) Let $r, b \geq 1$. Prove that there exists a number $R(r,b)$ depending only on r and b with the following property: for every complete graph G with $R(r,b)$ vertices whose edges are coloured red or blue, there exists either a complete subgraph on r vertices which is entirely red, or a complete subgraph on b vertices which is entirely blue.

Problem 6. (15 points)

(a) Let $f : \{0, 1, \ldots, n\} \to \{0, 1\}$. Prove that there exists the unique polynomial $W : \mathbb{R} \to \mathbb{R}$ with $\deg(W) \le n$ such that $W(k) = f(k)$ for $0 \le k \le n$. Prove that $\deg(W) = 0$ or $\deg(W) \ge n/2$.

(b) Let $f : \{0, 1, \ldots, n\} \to \mathbb{R}$ and let us consider the unique polynomial $W : \mathbb{R} \to \mathbb{R}$ with $\deg(W) \le n$ such that $W(k) = f(k)$ for $0 \le k \le n$. Then for $0 \le r \le n$ the following are equivalent

  (i) $\deg(W) \le n - r$,

  (ii) for $n - r < m \le n$ we have $\sum_{0 \le j \le m} (-1)^j \binom{m}{j} f(j) = 0$.

Problem 7. (5 points) Let $x_1, x_2, \ldots, x_n$ be real numbers. Prove the identity

$$\max\{x_1, x_2, \ldots, x_n\} = \sum_{i=1}^{n} x_i - \sum_{i<j} \min\{x_i, x_j\} + \sum_{i<j<k} \min\{x_i, x_j, x_k\} - \ldots + (-1)^{n+1} \min\{x_1, x_2, \ldots, x_n\}.$$

Problem 8. (20 points) Let A be a symmetric matrix with eigenvalues $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$.

(a) Prove that for every $k = 1, 2, \ldots, n$ we have

$$\lambda_k = \max_{U:\ \dim(U)=n-k+1} \ \min_{x \in U,\ x \neq 0} \frac{\langle Ax, x\rangle}{\langle x, x\rangle} = \min_{U:\ \dim(U)=k} \ \max_{x \in U,\ x \neq 0} \frac{\langle Ax, x\rangle}{\langle x, x\rangle}.$$

In particular

$$\lambda_1 = \min_{x \neq 0} \frac{\langle Ax, x\rangle}{\langle x, x\rangle}, \qquad \lambda_n = \max_{x \neq 0} \frac{\langle Ax, x\rangle}{\langle x, x\rangle}.$$

(b) Let $A = (a_{ij})_{i,j=1}^{n}$ be a symmetric matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$. Prove that $\sum_{i,j=1}^{n} a_{ij}^2 = \sum_{i=1}^{n} \lambda_i^2$.

(c) We define the operator norm and the Hilbert-Schmidt norm of a real $n \times n$ matrix $A = (a_{ij})$,

$$\|A\| = \sup_{x \in \mathbb{R}^n:\ x \neq 0} \frac{|Ax|}{|x|}, \qquad \|A\|_{HS} = \left(\sum_{ij} a_{ij}^2 \right)^{1/2}.$$

Prove that $\|A\|^2$ is the maximal eigenvalue of the matrix $A^T A$ and AA^T. Deduce that is the case of symmetric matrices we have $\|A\| = \max_i |\lambda_i|$. Prove that $\|A\| \le \|A\|_{HS}$.

(d) Let $n \geq 2$ and let $a_{ij} \in \{-1, 1\}$ for $1 \leq i < j \leq n$. Prove that there exists a vector $x \in \mathbb{R}^n$ with $|x| = 1$ such that $\left|\sum_{1\leq i<j\leq n}^{n} a_{ij} x_i x_j\right| \geq c\sqrt{n}$.

Problem 9. (10 points) We say that a polygon P (a subset of a plane bounded by a piecewise linear curve without self-intersections) has an ear at a vertex V if the line $V_- V_+$, where V_-, V_+ are adjacent to V lies entirely inside the polygon P. Two ears are said to be non-overlapping if the interiors of triangles VV_-V_+ are disjoint.

(a) Prove that except for triangles, every polygon has at least two non-overlapping ears.

(b) Prove that there exists a triangulation of P with no additional vertices and a 3-coloring of the vertices of P such that each triangle does not have two vertices with the same colour.

(c) The art gallery has a shape of an polygon P with n-vertices. Show that one can place $[n/3]$ guards in vertices of P who together can observe the whole gallery.

Z matki obcej; krew jego dawne bohater,
A imię jego będzie czterdzieści i cztery.
Adam Mickiewicz, Dziady[1]

Selected theorems in mathematics

Part III, prepared by: Piotr Nayar

Problem 1. (20 points) Let $x_1, x_2, \ldots, x_n$ be vectors in a Euclidean space $(\mathbb{R}^N, \|\cdot\|)$ and let $2 \leq k \leq n$. Prove the inequality

$$\binom{n-2}{k-2}\left(\sum_{i=1}^{n}\|x_i\| - \left\|\sum_{i=1}^{n} x_i\right\|\right) \leq \sum_{1\leq i_1<i_2<\cdots<i_k\leq n}\left(\|x_{i_1}\| + \cdots + \|x_{i_k}\| - \|x_{i_1} + \cdots + x_{i_k}\|\right).$$

In particular, prove that if x, y, z are vectors in a Euclidean space $(\mathbb{R}^N, \|\cdot\|)$ then we have

$$\|x+y\| + \|y+z\| + \|z+x\| \leq \|x\| + \|y\| + \|z\| + \|x+y+z\|.$$

Problem 2. (20 points) Let $z_1, z_2, \ldots, z_n$ be complex numbers. Prove that there exists a subset I of $\{1, 2, \ldots, n\}$ such that

$$\left|\sum_{k\in I} z_k\right| \geq \frac{1}{\pi}\sum_{k=1}^{n}|z_k|.$$

Is the constant $1/\pi$ optimal?

Problem 3. (20 points) Consider a $n \times m$ matrix A with $0, 1$ entries. We assume that the number of 1's in the matrix A equals $2j$, where j is an integer. Is it always possible to remove some number of columns and rows of A is such a way that the number of 1's in the remaining matrix is j?

Problem 4. (30 points)

(a) Let A and B be non-empty compact sets in $\mathbb{R}$. Prove that for every $\lambda \in [0, 1]$ we have

$$|\lambda A + (1-\lambda)B| \geq (1-\lambda)|A| + \lambda|B|.$$

(b) Let f, g and m be nonnegative measurable functions on $\mathbb{R}$ and let $\lambda \in [0, 1]$. Assume that for all $x, y \in \mathbb{R}$ we have

$$m((1-\lambda)x + \lambda y) \geq f(x)^{1-\lambda} g(y)^{\lambda}.$$

Prove that

$$\int_{\mathbb{R}} m \geq \left(\int_{\mathbb{R}} f\right)^{1-\lambda}\left(\int_{\mathbb{R}} g\right)^{\lambda}. \tag{1}$$

(c) Prove the inequality (1) in $\mathbb{R}^n$.

Problem 5. (15 points)

(a) Prove that any sequence of real numbers $x_1, x_2, \ldots$ contains a non-increasing or a non-decreasing subsequence.

(b) Let $n, m \geq 1$ be integers. Suppose we have a sequence of $(n-1)(m-1)+1$ real numbers. Prove that there exists a non-decreasing sequence of length n or a non-increasing sequence of length m.

[1] *Born from a foreign mother, his blood of ancient heroes, And his name will be forty and four*, Adam Mickiewicz, *Forefathers.*

Selected theorems in mathematics

Part IV, prepared by: Piotr Nayar

Problem 1. (15 points) Let $A = (a_{ij})_{i,j=1}^n$ be a skew-symmetric real matrix, i.e. $A^T = -A$. Prove that there exists a polynomial P in variables a_{ij} such that $\det(A) = P^2$.

Problem 2. (15 points)

(a) Prove the Brunn-Minkowski inequality, which states that if A and B are non-empty compact sets then for all $\lambda \in [0,1]$ we have

$$|(1-\lambda)A + \lambda B| \geq |A|^{1-\lambda}|B|^{\lambda}$$

and

$$|(1-\lambda)A + \lambda B|^{1/n} \geq (1-\lambda)|A|^{1/n} + \lambda|B|^{1/n}.$$

(b) Prove the isoperimetric inequality, i.e., show that when $|A| = |B|$, where A is a measurable set in $\mathbb{R}^n$ and B is an Euclidean ball in $\mathbb{R}^n$, then $|A_t| \geq |B_t|$, where $A_t = \{x \in \mathbb{R}^n,\ \mathrm{dist}(x, A) \leq t\}$.

(c) Let A be a compact subset of $\mathbb{R}^n$ and let us define

$$|\partial A| = \liminf_{t \to 0^+} \frac{|A + tB_2^n| - |A|}{t},$$

where B_2^n is an Euclidean ball. Show that the condition $|A| = |B|$, where B is a Euclidean ball in $\mathbb{R}^n$ implies $|\partial A| \geq |\partial B|$.

Problem 3. (10 points) Fix $1 \leq k \leq n$. Let $A_1, A_2, \ldots, A_m$ be distinct subsets of $\{1, 2, \ldots, n\}$ such that $|A_i \cap A_j| = k$ for all $i \neq j$. Prove that $m \leq n$.

Problem 4. (10 points) Suppose that G is a graph on n vertices with more than $n^2/4$ edges. Prove that G contains a triangle. Show that for an even number n there exists a graph G with n vertices and $n^2/4$ edges containing no triangle.

Problem 5. (20 points) Let X, Y be independent identically distributed real random variables. Prove that

$$\mathbb{E}|X+Y| \geq \mathbb{E}|X-Y|.$$

Problem 6. (20 points) Let $(\mathbb{Z}^d, E)$ be an integer lattice, i.e., a graph such that $\{x, y\} \in E$ if and only if $|x-y| = 1$. A path from x_0 to x_n is a sequence $x_0, x_1, \ldots, x_n \in \mathbb{Z}^d$ such that $\{x_i, x_{i+1}\} \in E$ for $i = 0, 1, \ldots, n-1$. Such a path is called a path of length n from x_0 to x_n. For $u, v \in \mathbb{Z}^d$ let $P^k(u,v)$ be the number of paths from u to v having length k. Prove that for every $k \geq 1$ and every $u, v \in \mathbb{Z}^d$ we have $P^{2k}(u,u) \geq P^{2k}(u,v)$.

Selected theorems in mathematics

Part V, prepared by: Piotr Nayar

Problem 1. (10 points) Let $f : \mathbb{R}^n \to \mathbb{R}$ be C^∞. Prove that there exist smooth functions $g_1, \ldots, g_n$ such that $g_i(0) = \frac{\partial f}{\partial x_i}(0)$ and

$$f(x) = f(0) + \sum_{i=1}^{n} x_i g_i(x).$$

Problem 2. (20 points) Let $c > 0$ be a real number. Prove the inequalities

$$\frac{1}{c^2 + \frac{1}{2}} < \sum_{n=1}^{n} \frac{2n}{(n^2 + c^2)^2} < \frac{1}{c^2}.$$

Problem 3. (15 points) Let A be a complex $n \times n$ matrix. Prove that the following conditions are equivalent,

(a) A is nilpotent, i.e., there exists $p \geq 1$ such that $A^p = 0$,

(b) $A^n = 0$,

(c) the characteristic polynomial of A is equal to λ^n,

(d) all the eigenvalues of A are 0,

(e) $\mathrm{tr}(A^p) = 0$ for $p = 1, \ldots, n$.

Problem 4. (10 points) Let $t(n) = |\{1 \leq k \leq n : k|n\}|$. Prove that for $n \geq 1$ we have

$$\left| \frac{t(1) + t(2) + \ldots + t(n)}{n} - \ln n \right| \leq 1.$$

Problem 5. (20 points) Let $G = (V, E)$ be a graph. The choice number $\mathrm{ch}(G)$ is the minimal number k such that for every assignment of a set $S(v)$ of k colors to every vertex v of G, there is a choice $k_v \in S(v)$ of colors such that $\{u, v\} \in E$ implies $k_v \neq k_u$.

Prove that for every bipartite $n \times n$ graph with $n \geq 3$ we have $\mathrm{ch}(G) \leq 2 \log_2 n$. Show that this bound is optimal, up to the multiplicative constant.

Problem 6. (15 points)

(a) Let $a_0 \geq a_1 \geq a_2 \geq \ldots \geq a_n > 0$ be real numbers and consider the polynomial $P : \mathbb{C} \to \mathbb{C}$ given by $P(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0$. Suppose that $P(z_0) = 0$ for some $z_0 \in \mathbb{C}$. Prove that $|z_0| \geq 1$.

(b) Let $a_0, \ldots, a_n > 0$. Then all the zeros of $P(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0$ lie in the annulus

$$r := \min_{k=0,1,\ldots,n-1} \frac{a_k}{a_{k+1}} \leq |z| \leq \max_{k=0,1,\ldots,n-1} \frac{a_k}{a_{k+1}} =: R.$$

Selected theorems in mathematics

Part VI, prepared by: Piotr Nayar

Problem 1. (10 points) Let $x_1, \ldots, x_n, y_1, \ldots, y_n$ be real numbers such that $x_i + y_j \neq 0$ for $i, j = 1, \ldots, n$. Let $A = (a_{ij})_{i,j=1}^n$, where $a_{ij} = (x_i + y_j)^{-1}$. Prove that

$$\det(A) = \frac{\prod_{j>i}(x_j - x_i)(y_j - y_i)}{\prod_{i,j}(x_i + y_j)}.$$

Problem 2. (20 points) Let $\mathbb{F}$ be a finite field. A polynomial $P \in \mathbb{F}[x_1, \ldots, x_n]$ over $\mathbb{F}$ is a finite formal expression of the form

$$P(x_1, \ldots, x_n) = \sum_{i_1, \ldots, i_n \geq 0} c_{i_1, \ldots, i_n} x^{i_1} \cdot \ldots \cdot x_n^{i_n}.$$

We define the set

$$Z(P)[F] = \{(x_1, \ldots, x_n) \in \mathbb{F}^n : \; P(x_1, \ldots, x_n) = 0\}.$$

(a) Show that if $E \subset \mathbb{F}^n$ has cardinality less than $\binom{d+n}{n}$, then there exist a non-zero polynomial $P \in \mathbb{F}[x_1, \ldots, x_n]$ of degree at most d such that $E \subset Z(P)[F]$.

(b) Show that if $P \in \mathbb{F}[x_1, \ldots, x_n]$ is a non-zero polynomial of degree at most d, then we have $|Z(P)[F]| \leq d|\mathbb{F}|^{n-1}$. Show that if P, regarded as a function $P : \mathbb{F}^n \to \mathbb{F}$, vanishes on $\mathbb{F}^n$, then $\deg(P) \geq |\mathbb{F}|$.

Problem 3. (15 points) A family $\mathcal{A}$ of subsets of $[n] = \{1, \ldots, n\}$ is called monotone if $B \in \mathcal{A}$ implies $C \in \mathcal{A}$ for any set $C \subset B$. Prove that if $\mathcal{A}$ and $\mathcal{B}$ are monotone families of subsets of $[n]$ then we have

$$2^n |\mathcal{A} \cap \mathcal{B}| \geq |\mathcal{A}| \cdot |\mathcal{B}|.$$

Problem 4. (10 points) Let X be the random vector uniformly distributed on the cube $[-\sqrt{3}, \sqrt{3}]^n$. Prove that

$$\mathbb{E}\left(|X| - \sqrt{n}\right)^2 \leq 1.$$

Problem 5. (15 points) Let $x_1, x_2, \ldots, x_n$ be a non-increasing sequence of positive real numbers. Prove the inequality[1]

$$\sum_{i=1}^{n-1} \frac{1}{\sqrt{i}} \sqrt{\sum_{j=i+1}^{n} x_j^2} < \frac{\pi}{2} \sum_{i=1}^{n} x_i.$$

[1] On the left hand side there is no x_1, this is not a mistake.

Selected theorems in mathematics

Part VII, prepared by: Piotr Nayar

Problem 1. (20 points) Let C be a smooth closed curve on the unit sphere S^2 of length less than 2π. Prove that this curve is contained in a certain open hemisphere.

Problem 2. (10 points) Let A be a measurable set on S^1 with $|A| = \pi$. Prove that there exists a complex number z with $|z| = 1$ such that $|A \cup (zA)| \geq \frac{3}{2}\pi$.

Problem 3. (20 points) Let $u_1, u_2, \ldots, u_m$ be non-zero vectors in the Euclidean space $\mathbb{R}^n$ satisfying the condition $\langle u_i, u_j \rangle \leq 0$ for all $i \neq j$.

(a) Prove that if $\sum_{i \in I} \alpha_i u_i = 0$ for some $I \subset [m]$ and non-zero $\alpha_i, i \in I$ and for every $J \subset I$, $J \neq I$ we have $\sum_{i \in J} \alpha_i u_i \neq 0$ then all the numbers α_i have the same sign.

(b) Prove that $m \leq 2n$.

(c) Let $d \geq 1$ be an integer. Prove that $C_1, \ldots, C_m$ are binary vectors of length $2d$ such that for all $i \neq j$ the vectors C_i and C_j have different bits on at least d coordinates, then $m \leq 4d$.

Problem 4. (15 points) Let $f(x) = \sum_{k=n}^{m} a_n \sin(kx)$. Prove that f has at least $2n$ zeros in the interval $[0, 2\pi)$.

Problem 5. (15 points) Let $2k \leq n$ and let $\mathcal{A}$ be a family of subsets of $[n]$ such that each subset has size k and for every $A, B \in \mathcal{A}$ we have $A \cap B \neq \emptyset$. Prove that $|\mathcal{A}| \leq \binom{n-1}{k-1}$.

Selected theorems in mathematics

Part VIII, prepared by: Piotr Nayar

Problem 1. (15 points) Suppose that a probability measure μ on $[0,\infty)$ is absolutely continuous with respect to the Lebesgue measure. Let $T(x)=\mu([x,\infty))$. Prove that for every non-decreasing function $g:[0,\infty)\to(0,\infty)$ we have

$$\mathrm{Ent}_\mu(g) \le -\int_0^\infty g(x)\,(1+\ln T(x))\,\mathrm{d}\mu(x),$$

where $\mathrm{Ent}_\mu(g)=\int_0^\infty g\ln g\,\mathrm{d}\mu - \left(\int_0^\infty g\,\mathrm{d}\mu\right)\ln\left(\int_0^\infty g\,\mathrm{d}\mu\right)$.

Problem 2. (15 points) A family $\mathcal{F}$ of subsets of $[n]=\{1,\ldots,n\}$ shatters a set $S\subseteq[n]$ if for every $R\subseteq S$ there is $F\in\mathcal{F}$ such that $S\cap F=R$. Prove that if

$$|\mathcal{F}| > \binom{n}{0}+\binom{n}{1}+\ldots+\binom{n}{k}$$

then there exists a set $S\subseteq[n]$ of cardinality $k+1$ such that $\mathcal{F}$ shatters S.

Problem 3. (15 points) Let $\mathbb{F}$ be a finite field with q elements and let $n\ge 1$. Let $N\subseteq\mathbb{F}^n$ be a subset such that for every $x\in\mathbb{F}^n$ there exists $v\in\mathbb{F}^n$ for which the line $L(x)=\{x+vt: t\in\mathbb{F}\}$ satisfies $|L(x)\cap N|\ge q/2$. Prove that $|N|\ge c_n q^n$, where c_n depends only on n.

Problem 4. (15 points) Let $c_1,c_2,\ldots,c_n\in\mathbb{C}$. Find the eigenvalues and eigenvectors of the matrix

$$A=\begin{bmatrix} c_1 & c_2 & \cdots & c_n \\ c_n & c_1 & \cdots & c_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ c_2 & c_3 & \cdots & c_1 \end{bmatrix}.$$

Problem 5. (10 points) Let $|\cdot|$ be the standard Euclidean norm and let $v_1,\ldots,v_n\in\{-1,1\}^n$. Prove that there exist $\varepsilon_1,\ldots,\varepsilon_n\in\{-1,1\}$ such that

$$\left|\sum_{i=1}^n \varepsilon_i v_i\right| \le n.$$

Selected theorems in mathematics

Part IX, prepared by: Piotr Nayar

Problem 1. (15 points) Let K be a convex compact set in $\mathbb{R}^n$, where $n \geq 2$. Take $\theta \in S^{n-1}$ and define $H_r = \{x \in \mathbb{R}^n, \langle x, \theta \rangle = r\}$. Prove that the function

$$r \mapsto (\mathrm{vol}(K \cap H_r))^{1/(n-1)}$$

is concave on its support.

Problem 2. (50 points) Consider $n \times n$ matrices with independent symmetric ± 1 entries. Prove that

$$\lim_{n\to\infty} \mathbb{P}(\det M_n = 0) = 0.$$

Hint.

(a) Let $X_1, \ldots, X_n$ be the rows of M_n. Observe that

$$\mathbb{P}(\det M_n = 0) \leq \sum_{i=1}^{n-1} \mathbb{P}\left(X_{i+1} \in \mathrm{span}(X_1, \ldots, X_i)\right).$$

(b) Prove that every d dimensional subspace of $\mathbb{R}^n$ contains at most 2^d vectors with ± 1 entries. Deduce that

$$\mathbb{P}\left(X_{i+1} \in \mathrm{span}(X_1, \ldots, X_i)\right) \leq \frac{2^i}{2^n}, \qquad i = 1, \ldots, n-1.$$

(c) We say that a $n \times n$ matrix is l-universal if for any set of l indices $i_1, \ldots i_l$ and any set of signs $\varepsilon_1, \ldots, \varepsilon_l$, there is a row X where the i_j-th entry of X has sign ε_j , for all $1 \leq j \leq l$. Prove that the probability that M_n is not l-universal is less then $\binom{n}{l} 2^l (1 - 2^{-l})^n$.

(d) Show that if M_n is l-universal then any vector v orthogonal to $X_1, \ldots, X_n$ must have at least l non-zero coordinates. Moreover, prove that if v is a vector with l non-zero coordinates then $\mathbb{P}\left(X_n \cdot v = 0\right) \leq C_1/\sqrt{l}$, where $C_1 > 0$ is a universal constant.

(e) Prove that

$$\mathbb{P}\left(X_n \in \operatorname{span}(X_1, \ldots, X_{n-1})\right) \leq \frac{C_2}{\ln^{1/2} n}.$$

(f) Divide the sum into two parts,

$$\sum_{i=1}^{n-1} \mathbb{P}\left(X_{i+1} \in \operatorname{span}(X_1, \ldots, X_i)\right) = \sum_{i=1}^{k-1} \mathbb{P}\left(X_{i+1} \in \operatorname{span}(X_1, \ldots, X_i)\right) + \sum_{i=k}^{n-1} \mathbb{P}\left(X_{i+1} \in \operatorname{span}(X_1, \ldots, X_i)\right).$$

Threat the first sum as in point (b) and the second sum as in points (c)-(e).

Problem 3. (20 points)

(a) Consider a function $f : \{-1,1\}^n \to \{-1,1\}$. For $x = (x_1, \ldots, x_n) \in \{-1,1\}^n$ define $x^i = (x_1, \ldots, -x_i, \ldots, x_n)$. Let μ be a uniform probability measure on $\{-1,1\}^n$. Prove the Poincaré inequality

$$\operatorname{Var}_\mu(f) \leq \int \sum_{i=1}^{n} (f(x) - f(x^i))^2 \, \mathrm{d}\mu$$

(b) Let $A \subseteq \{-1,1\}^n$. We write $x \sim y$ when $y = x^i$ for some $i = 1, \ldots, n$. Define the edge boundary of A,

$$\partial A = \{(x, y) \in \{-1,1\}^n, x \sim y, x \in A, y \notin A\}.$$

Prove the isoperimetric type inequality

$$|\partial A| \geq 2^{n+1} \mu(A)(1 - \mu(A)).$$

Selected theorems in mathematics

Part IX, prepared by: Piotr Nayar

Problem 1. (10 points) For a function $f : \{-1,1\}^n \to \mathbb{R}$ with a Walsh-Fourier expansion $f = \sum_{S\subseteq[n]} a_s w_s$ and $\delta \in [0,1]$ we define

$$T_\delta(f) = \sum_{S\subseteq[n]} a_s \delta^{|S|} w_s.$$

Prove that

$$T_\delta(f)(x_1,\ldots,x_n) = \int_{\{-1,1\}^n} f(y_1,\ldots,y_n)K(x_1,y_1)\ldots K(x_n,y_n)\ \mathrm{d}\mu(y_1)...\mathrm{d}\mu(y_n),$$

where $K(x,y) = 1 + \delta yx$ and μ is a uniform measure on $\{-1,1\}$.

Problem 2. (20 points) Let $q \geq p \geq 1$ and let (Ω_1,μ_1), (Ω_2,μ_2) be two finite probability spaces. Let $K_i : \Omega_i \times \Omega_i \to \mathbb{R}$ for $i = 1,2$. We define two operators

$$T_i(f)(x) = \int_{\Omega_i} K_i(x,y)\ \mathrm{d}\mu_i(y), \qquad i = 1,2.$$

Moreover, for $f : \Omega_1 \times \Omega_2 \to \mathbb{R}$ let us take

$$(T_1 \otimes T_2)(f)(x_1,x_2) = \int_{\Omega_1}\int_{\Omega_2} f(y_1,y_2)K_1(x_1,y_1)K_2(x_2,y_2)\ \mathrm{d}\mu_2(y_2)\ \mathrm{d}\mu_1(y_1).$$

Suppose that for $i = 1,2$ we have

$$\|T_i f\|_{L_q(\Omega_i,\mu_i)} \leq \|f\|_{L_p(\Omega_i,\mu_i)}, \qquad \text{for all } \ f : \Omega_i \to \mathbb{R}.$$

Then

$$\|(T_1 \otimes T_2)f\|_{L_q(\Omega_1\times\Omega_2,\mu_1\otimes\mu_2)} \leq \|f\|_{L_p(\Omega_1\times\Omega_2,\mu_1\otimes\mu_2)}.$$

Problem 3. (30 points) Prove that for any $f : \{-1,1\}^n \to \mathbb{R}$ and any $\delta \in [0,1]$ we have

$$\|T_\delta f\|_2 \leq \|f\|_{1+\delta_2}.$$

Selected theorems in mathematics

Part I, prepared by Piotr Nayar

Problem 1. (15 points)

(a) Let $\mathbb{F}$ be an arbitrary field and let $P(x_1, \ldots, x_n)$ be a polynomial in . Suppose that the degree of P is $\sum_{i=1}^n k_i$, where each k_i is a non-negative integer and suppo$\mathbb{F}[x_1, \ldots, x_n]$se that the coefficient of $x_1^{k_1} \cdot \ldots \cdot x_n^{k_n}$ is non-zero. Then for any subsets $A_1, \ldots, A_n$ of $\mathbb{F}$ satisfying $|A_i| \geq k_i + 1$ for all $i = 1, \ldots, n$, there exist $a_1 \in A_1, \ldots, a_n \in A_n$ such that $P(a_1, \ldots, a_n) \neq 0$.

(b) Suppose that the hyperplanes $H_1, \ldots, H_m \subset \mathbb{R}^n$ cover the set $\{0,1\}^n \backslash \{0\}$ and that $0 \notin \bigcup_{i=1}^m H_i$. Prove that $m \geq n$.

Solution. a) This theorem is called the Combinatorial Nullstellensatz and was proved by Noga Alon in [A]. The proof is taken from [M]. We proceed by induction on $\deg(P)$. If $\deg(P) = 1$ then our assertion is trivial. Suppose that $\deg(P) > 1$ and P satisfies the assumptions of the theorem but the assertion is false, that is $P(x) = 0$ for every $x \in A_1 \times \cdots \times A_n$. Without loss of generality we assume that $k_1 > 0$. Fix $a \in A_1$. There exist polynomials $Q \in \mathbb{F}[x_1, \ldots, x_n]$ and $R \in \mathbb{F}[x_2, \ldots, x_n]$ such that

$$P = (x_1 - a)Q + R. \tag{1}$$

Note that $\deg(Q) = \deg(P) - 1$ and that Q has a non-vanishing monomial of the form $x_1^{k_1 - 1} x_2^{k_2} \cdot \cdots \cdot x_n^{k_n}$. Take any $x \in \{a\} \times A_2 \times \cdots \times A_n$. Since $P(x) = 0$ we obtain $R(x) = 0$. However, R does not contain x_1, thus $R(x) = 0$ for all $x \in (A_1 \backslash \{a\}) \times A_2 \times \cdots \times A_n$. Take such an x and substitute it to (1). Since $x_1 - a$ is non-zero and $P(x) = R(x) = 0$ we obtain $Q(x) = 0$. So, $\deg(Q) = \deg(P) - 1$, Q contains a monomial $x_1^{k_1 - 1} x_2^{k_2} \cdot \cdots \cdot x_n^{k_n}$ and Q vanishes on the set $(A_1 \backslash \{a\}) \times A_2 \times \cdots \times A_n$, where $|A_1 \backslash \{a\}| \geq k_i, |A_2| \geq k_2 + 1, \ldots, |A_n| \geq k_n + 1$. This contradicts the inductive assumption.

b) The solution is take from [A]. Suppose that the hyperplane H_i is given by the equation $< a_i, x >= b_i$. We have $b_i \neq 0$ since H_i does not cover the origin. Assume that our assertion is false and $m < n$. Define the following polynomial,

$$P(x) = (-1)^{n+m+1} \prod_{j=1}^m b_j \prod_{i=1}^n (x_i - 1) + \prod_{i=1}^m (< a_i, x > - b_i) \,.$$

The degree of this polynomial is n and the coefficient of $\prod_{i=1}^n x_i$ is $(-1)^{n+m+1}\prod_{j=1}^m b_j \neq 0$. Therefore, from part a) there exists $x_0 \in \{0,1\}^n$ such that $P(x_0) \neq 0$. This point is not the origin since clearly $P(0) = 0$. Therefore, on x_0 the polynomial $\prod_{j=1}^n (x_i - 1)$ vanishes and

$$P(x_0) = \prod_{i=1}^m (< a_i, x_0 > -b_i) \neq 0.$$

It means that $< a_i, x_0 > \neq b_i$ for all $i = 1, \ldots, m$ and therefore $x_0 \notin \bigcup_{i=1}^m H_i$. This is a contradiction. $\square$

Problem 2. (15 points)

(a) Let f be a trigonometric polynomial of order n, i.e.,

$$f(x) = \sum_{k=0}^n (a_k \cos kx + b_k \sin kx).$$

Here the coefficients $(a_k), (b_k)$ can be complex. Let us define the function $D_n(x) = \frac{1}{2} + \sum_{k=1}^{n-1} \cos kx + \frac{1}{2}\cos nx$ and the set $A_n = \{\frac{2k-1}{2n}\pi,\ k = 1, 2, \ldots, 2n\}$. Prove the identity

$$f(x) = a_n \cos nx + \frac{1}{n} \sum_{t \in A_n} f(t) D_n(x - t). \tag{2}$$

(b) Prove that $D_n(x) = \frac{\sin nx}{2\mathrm{tg}(x/2)}$. Prove the identities

$$f'(0) = \frac{1}{n} \sum_{t \in A_n} f(t) \frac{(-1)^{k+1}}{(2\sin(t/2))^2}, \qquad \sum_{t \in A_n} \frac{1}{(2\sin(t/2))^2} = n^2$$

and deduce that

$$f'(x) = \frac{1}{n} \sum_{t \in A_n} f(x+t) \frac{(-1)^{k+1}}{(2\sin(t/2))^2}. \tag{3}$$

(c) Show that for every non-decreasing convex function $\phi : [0, \infty) \to \mathbb{R}$ we have

$$\int_0^{2\pi} \phi\left(\left|\frac{f'(x)}{n}\right|\right) \mathrm{d}x \leq \int_0^{2\pi} \phi(|f(x)|) \, \mathrm{d}x. \tag{4}$$

Deduce that for $1 \leq p < \infty$ we have

$$\left(\int_0^{2\pi} |f'(x)|^p \, \mathrm{d}x\right)^{1/p} \leq n \left(\int_0^{2\pi} |f(x)|^p \, \mathrm{d}x\right)^{1/p}. \tag{5}$$

Moreover,

$$\max_{x\in[0,2\pi]} |f'(x)| \le n \max_{x\in[0,2\pi]} |f(x)|. \tag{6}$$

Solution. The inequality (4) is the so-called Zygmund inequality, see [Z]. The inequality (6) is the classical Bernstein inequality.

(a) Our identity is linear. Therefore, it suffices to check it for functions $f(x) = e^{ikx}$, $k = -n, \dots, n$. Note that D_n is a sum of functions of this form. Let us compute $\sum_{t\in A_n} e^{ikt}e^{il(x-t)}$, where $l = -n, \dots n$ and $k = -n, \dots, n$. If $|k| < n$ and $k \neq l$ then

$$\sum_{t\in A_n} e^{ikt}e^{il(x-t)} = e^{ilx}\sum_{k=1}^{2n} e^{i(k-l)\frac{2k-1}{2n}\pi} = e^{ilx}e^{-i(k-l)\frac{\pi}{2n}}\sum_{k=0}^{2n-1} e^{i(k-l)\frac{2k}{2n}\pi}$$
$$= e^{ilx}e^{-i(k-l)\frac{\pi}{2n}}\frac{e^{2i(k-l)\pi}-1}{e^{i(k-l)\frac{\pi}{n}}-1} = 0.$$

Note that in this case $e^{i(k-l)\frac{\pi}{n}} \neq 1$. If $k = l$ then we have

$$\sum_{t\in A_n} e^{ikt}e^{il(x-t)} = 2ne^{ikx},$$

It follows that for $|k| < n$ we have

$$\frac{1}{n}\sum_{t\in A_n} e^{ikt}D_n(x-t) = \frac{1}{2n}\cdot 2ne^{ikx} = e^{ikx},$$

where we have use the equality $\cos kx = \frac{e^{ikx}+e^{-ikx}}{2}$.

Now we are left with the cases $f(x) = e^{inx}$ and $f(x) = e^{-inx}$. Note that (2) is invariant under conjugation. Therefore, we only need to consider $f(x) = e^{inx}$. Note that $e^{inx} = \cos nx + i\sin nx$. Thus, $a_n = 1$. The expression

$$\sum_{t\in A_n} e^{int}e^{il(x-t)}$$

is non-zero only when $|l| = n$ and it is equal to $2ne^{inx}$ if $l = n$. If $l = -n$ then

$$\sum_{t\in A_n} e^{int}e^{il(x-t)} = e^{-inx}\sum_{t\in A_n} e^{2int} = e^{-inx}\sum_{t\in A_n} e^{i(2k-1)\pi} = -2ne^{-inx}.$$

Thus,

$$a_n\cos nx + \frac{1}{n}\sum_{t\in A_n} f(t)D_n(x-t) = \frac{e^{inx}+e^{-inx}}{2}\frac{1}{n}\sum_{t\in A_n} f(t)\frac{1}{2}\cos(n(x-t))$$
$$= \frac{e^{inx}+e^{-inx}}{2} + \frac{1}{4n}\left(2ne^{inx} - 2ne^{-inx}\right) = e^{inx}.$$

(b) One can compute $D_n(x)$ easily by using the indentity

$$\cos kx = \frac{\sin(kx + \frac{x}{2}) - \sin(kx - \frac{x}{2})}{2\sin(x/2)}, \quad x/2\pi \notin \mathbb{Z}.$$

The identity

$$f'(0) = \frac{1}{n} \sum_{t \in A_n} f(t) \frac{(-1)^{k+1}}{(2\sin(t/2))^2} \tag{7}$$

follows by differentiating (2). To check that $\sum_{t\in A_n} \frac{1}{(2\sin(t/2))^2} = n^2$ it suffices to take $f(x) = \sin nx$. To obtain (3) take $t \mapsto f(t+x)$ instead of $t \mapsto f(t)$ in (7).

(c) From (b) we have the identity of the form

$$\frac{f'(x)}{n} = \sum_{t \in A_n} \lambda_t f(x+t),$$

where $\sum_{t\in A_n} \lambda_t = 1$. Using Jensen's inequality and the monotonicity of ϕ we obtain

$$\begin{aligned} \int_0^{2\pi} \phi\left(\left|\frac{f'(x)}{n}\right|\right) \mathrm{d}x &\le \int_0^{2\pi} \phi\left(\sum_{t\in A_n} |\lambda_t||f(x+t)|\right) \mathrm{d}x \\ &\le \sum_{t\in A_n} \lambda_t \int_0^{2\pi} \phi\left(|f(x+t)|\right) \mathrm{d}x = \sum_{t\in A_n} \lambda_t \int_0^{2\pi} \phi\left(|f(x)|\right) \mathrm{d}x \\ &= \int_0^{2\pi} \phi\left(|f(x)|\right) \mathrm{d}x. \end{aligned}$$

To obtain (5) it suffices to take $\phi(x) = x^p$. Now (6) follows by taking $p \to \infty$. $\square$

Problem 3. (10 points)

(a) Let $\mathcal{A}$ be a family of subsets of $\{1, 2, \ldots, n\}$ such that for any pair of subsets $A, B \in \mathcal{A}$ we have $A \not\subseteq B$. Prove that

$$|\mathcal{A}| \le \binom{n}{[n/2]}$$

and determine the extremal case.

(b) Let $v_1, \ldots, v_n$ be real numbers such that $|v_i| \ge 1$ for $i = 1, \ldots, n$. Define

$$A = \{x = (x_1, \ldots, x_n) \in \{-1, 1\}^n, \ \ |v_1x_1 + \cdots + v_nx_n| < 1\}.$$

Prove that $|A| \le \binom{n}{[n/2]}$.

Solution. (a) This is the so called Spencer's lemma, see [AI]. To prove the above fact we consider the family $\mathcal{A}$ and we count pairs (π, S), where π is a permutation of $\{1, \ldots, n\}$ and S is a set of the form $S = \{\pi(1), \ldots, \pi(k)\}$ for π, such that $S \in \mathcal{A}$. For each π we can have at most one $S \in \mathcal{A}$. Therefore, the number of pairs (π, S) is not greater than $n!$. Moreover, a fixed set $S \in \mathcal{A}$ of cardinality k will be counted exactly $k!(n-k)!$ times. So, if s_k is the number of sets in $\mathcal{A}$ of cardinality k then the number of pairs (π, S) is equal to $\sum_{k=0}^{n} s_k k!(n-k)!$. Thus,

$$\sum_{k=0}^{n} s_k k!(n-k)! \leq n!.$$

It means that

$$\frac{|\mathcal{A}|}{\binom{n}{[n/2]}} \leq \sum_{k=0}^{n} \frac{s_k}{\binom{n}{k}} \leq 1.$$

The family for which we have equality is the family of all subsets of cardinality $[n/2]$.

(b) This is a special case of the Littlewood-Offord problem, see [E]. Without loss of generality we can assume that $v_i \geq 1$ for $i = 1, \ldots, n$. A point x in $\{-1,1\}^n$ can be seen as a subset B_x of $\{1, 2, \ldots, n\}$, i.e., $i \in B_x$ if and only if $x_i = 1$. It is easy to observe that if $|v_1 x_1 + \cdots + v_n x_n| < 1$ for some $x \in \{-1,1\}^n$, then changing one or more signs x_i from -1 to 1 gives a point, for which $|v_1 x_1 + \cdots + v_n x_n| \geq 1$. It means that $\{B_x, x \in A\}$ satisfies the assumption of Spencer's lemma. Thus, $|A| \leq \binom{n}{[n/2]}$. $\square$

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Selected theorems in mathematics

Part II, prepared by: Piotr Nayar

Problem 1. (5 points) Let $A = (a_{ij})_{i,j=1}^n$ be a symmetric matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$. Prove that $\sum_{i,j=1}^n a_{ij}^2 = \sum_{i=1}^n \lambda_i^2$.

Solution. Note that $\sum_{ij} a_{ij}^2 = \mathrm{tr}(A^T A)$. Our assertion is clearly valid for diagonal matrices. If A is non-diagonal let us consider diagonal matrix D such that $D = C^T A C$, where C is orthogonal. The eigenvalues of D and A are the same. Moreover,

$$\begin{aligned}\sum_i \lambda_i^2 &= \mathrm{tr}(D^T D) = \mathrm{tr}((C^T A C)^T C^T A C) = \mathrm{tr}((C^T A^T C) C^T A C) \\ &= \mathrm{tr}(C^T A^T A C) = \mathrm{tr}(A^T A C C^T) = \mathrm{tr}(A^T A),\end{aligned}$$

where we have used the fact that $\mathrm{tr}(XY) = \mathrm{tr}(YX)$.

We can also show that $\sum_{ij} a_{ij}^2$ is invariant under rotations by a simple computation, i.e. we show that we have $\sum_{i,j} b_{ij}^2 = \sum_{i,j} a_{ij}^2$ whenever $A = (a_{ij})$, $B = (b_{ij})$ are matrices satisfying $B = C^T A C$, without using the above fact about the trace. Indeed,

$$\begin{aligned}\sum_{ij} b_{ij}^2 &= \sum_{ij} \left(\sum_{kl} c_{ki} a_{kl} c_{lj} \right)^2 = \sum_{ij} \sum_{k_1,k_2,l_1,l_2} c_{k_1 i} a_{k_1 l_1} c_{l_1 j} c_{k_2 i} a_{k_2 l_2} c_{l_2 j} \\ &= \sum_{k_1,k_2,l_1,l_2} a_{k_1 l_1} a_{k_2 l_2} \sum_{ij} c_{k_1 i} c_{l_1 j} c_{k_2 i} c_{l_2 j}.\end{aligned}$$

Now we observe that the rows of C are orthonormal. Therefore,

$$\sum_{ij} c_{k_1 i} c_{l_1 j} c_{k_2 i} c_{l_2 j} = \left(\sum_i c_{k_1 i} c_{k_2 i} \right) \left(\sum_j c_{l_1 j} c_{l_2 j} \right) = \delta_{k_1,k_2} \delta_{l_1,l_2}.$$

We arrive at

$$\sum_{ij} b_{ij}^2 = \sum_{k_1,k_2,l_1,l_2} a_{k_1 l_1} a_{k_2 l_2} \delta_{k_1,k_2} \delta_{l_1,l_2} = \sum_{kl} a_{kl}^2.$$

□

Problem 2. (10 points) Take $d \geq 1$ and let us consider $x_1, x_2, \dots, x_n \in \mathbb{R}^d$, where $n \geq d+2$. Prove that there exists a partition of $\{1, \dots, n\}$ into two sets I, J such that the convex hulls of the sets $\{x_i : \ i \in I\}$ and $\{x_j : \ j \in J\}$ have a nonempty intersection.

Solution. This is the so-called Radon's theorem. Without loss of generality we can assume that $n = d+2$. Note that $\{x_1 - x_{d+2}, \dots x_{d+1} - x_{d+2}\}$ is collection of $d+1$ vectors. Thus, these vectors are linear dependent, i.e. there exists a sequence of real numbers $a_1, \dots, a_{d+1}$ such that $\sum_{j=1}^{d+1} a_j(x_j - x_{d+2}) = 0$ and $a_{j_0} \neq 0$ for some j_0. Take $b_1 = a_1, \dots, b_{d+1} = a_{d+1}$ and $b_{d+2} = -(a_1 + \cdots + a_{d+1})$. It follows that $\sum_{j=1}^{d+2} b_j = 0$ and $\sum_{j=1}^{d+2} b_j x_j = 0$. The sets $I_+ = \{i : \ b_i > 0\}$, $I_- = \{i : \ b_i < 0\}$ are both nonempty and $\sum_{i \in I_+} b_i = \sum_{i \in I_-} (-b_i)$. Thus,

$$\frac{\sum_{i \in I_+} b_i x_i}{\sum_{i \in I_+} b_i} = \frac{\sum_{i \in I_-} (-b_i) x_i}{\sum_{i \in I_-} (-b_i)}.$$

The left hand side of the above equality belongs to $\mathrm{conv}\{x_i : i \in I_+\}$ while the right hand side is in $\mathrm{conv}\{x_i : i \in I_-\}$. $\square$

Problem 3. (10 points) Let $d \geq 1$ and let $A \subset \mathbb{R}^d$. Suppose $x \in \mathrm{conv}(A)$. Prove that there exists a set $B \subset A$ with $\#B \leq d+1$ such that $x \in \mathrm{conv}(B)$.

Solution. The above fact is the Carathéodory's theorem. It is easy to see that

$$\mathrm{conv}(A) = \left\{ \sum_{i=1}^{n} \lambda_i a_i : \ a_i \in A, \ \lambda_i \geq 0, \ \sum_{i=1}^{n} \lambda_i = 1, \ n \geq 1 \right\}.$$

Thus, we can write $x = \sum_{i=1}^{n} \lambda_i a_i$. If $n \leq d+1$ then there is nothing to prove. Assume that $n > d+1$. As in the solution of the Problem 2, there exists a sequence $\mu_1, \dots, \mu_n$ such that $\sum_{i=1}^{n} \mu_i = 0$ and $\sum_{i=1}^{n} \mu_i a_i = 0$ with $\mu_{j_0} \neq 0$ for some j_0. Thus, $x = \sum_{i=1}^{n} (\lambda_i - c\mu_i) a_i$ for every $c \in \mathbb{R}$. Take c such that $\lambda_i - c\mu_i \geq 0$, $1 \leq i \leq n$ and at least one such value is 0. We have expressed x as a convex combination of $n-1$ elements of A. We can further decrease the length of this sum as long as the condition $n > d+1$ is satisfied. $\square$

Problem 4. (20 points) Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function with period 1 and let α be irrational. Prove that

$$\lim_{n \to \infty} \frac{f(\alpha) + f(2\alpha) + \ldots + f(n\alpha)}{n} = \int_0^1 f(t) \, \mathrm{d}t. \tag{1}$$

Prove that for every interval $[a, b] \subset [0, 1]$ and every irrational real number α we have

$$\lim_{n\to\infty} \frac{\#\{1 \le k \le n : \ \{k\alpha\} \in [a, b]\}}{n} = b - a,$$

where $\{x\} \in [0, 1)$ is the fractional part of $x \in \mathbb{R}$.

Solution. The above theorem is the so-called Bohl-Sierpiński-Weil theorem. This is a version of ergodic theorem for the rotation on the circle. We first prove the above fact for the functions of the form $f(x) = e^{2\pi i k x}$, where $k \in \mathbb{Z}$. These functions are clearly 1-periodic. When $k = 0$ the assertion is obvious, thus we can assume that $k \neq 0$. Moreover, we have

$$\left|\frac{1}{n}\left(f(\alpha) + \ldots + f(n\alpha)\right)\right| = \left|\frac{e^{2\pi i\alpha}}{n} \cdot \frac{e^{2\pi i n\alpha} - 1}{e^{2\pi i\alpha} - 1}\right| \le \frac{1}{n} \cdot \frac{2}{|e^{2\pi i\alpha} - 1|} \xrightarrow[n\to\infty]{} 0.$$

Note that we have used the fact that $e^{2\pi i\alpha} - 1 \neq 0$ for $\alpha \notin \mathbb{Q}$. Since $\int_0^1 f(t)\ \mathrm{d}t = 0$, we obtain (1). By linearity the equality (1) is also true for every trigonometric polynomial, i.e., the function of the form $\sum_{k=-n}^{n} a_k e^{2\pi i k}$, where $a_k \in \mathbb{C}$ for $k = -n, \ldots, n$. From the Weierstrass theorem we know that these functions are dense in the space of all continuous complex-valued functions, i.e., for every continuous function $g : [0, 1] \to \mathbb{C}$ and for every $\varepsilon > 0$ there exists a trigonometric polynomial f such that $|g(t) - f(t)| \le \varepsilon$ for every $t \in [0, 1]$. If g is real then one can choose g to be real by taking the trigonometric polynomial $\Re g$ instead of g. Thus,

$$\begin{aligned}\limsup_{n\to\infty} \frac{1}{n}\left(g(\alpha) + \ldots + g(n\alpha)\right) &\le \varepsilon + \limsup_{n\to\infty} \frac{1}{n}\left(f(\alpha) + \ldots + f(n\alpha)\right) \\ &= \varepsilon + \int_0^1 f(t)\ \mathrm{d}t \le 2\varepsilon + \int_0^1 g(t)\ \mathrm{d}t.\end{aligned}$$

Taking $\varepsilon \to 0$ we obtain

$$\limsup_{n\to\infty} \frac{1}{n}\left(g(\alpha) + \ldots + g(n\alpha)\right) \le \int_0^1 g(t)\ \mathrm{d}t.$$

In the same way we show that

$$\liminf_{n\to\infty} \frac{1}{n}\left(g(\alpha) + \ldots + g(n\alpha)\right) \ge \int_0^1 g(t)\ \mathrm{d}t.$$

We have verified (1) for every continuous function.

To solve the second part let us take the characteristic function of the interval $[a, b] \subset [0, 1]$, extended periodically to the whole real line. Let us call this function f. Trivially, there exists a continuous function g such that $f(x) \leq g(x)$ and $\int_0^1 |f(t) - g(t)|\, \mathrm{d}t \leq \varepsilon$. Thus,

$$\begin{aligned}\limsup_{n\to\infty} \frac{1}{n}\left(f(\alpha) + \ldots + f(n\alpha)\right) &\leq \limsup_{n\to\infty} \frac{1}{n}\left(g(\alpha) + \ldots + g(n\alpha)\right) \\ &= \int_0^1 g(t)\, \mathrm{d}t \leq \varepsilon + \int_0^1 f(t)\, \mathrm{d}t.\end{aligned}$$

Taking $\varepsilon \to 0$ we obtain

$$\limsup_{n\to\infty} \frac{1}{n}\left(f(\alpha) + \ldots + f(n\alpha)\right) \leq \int_0^1 f(t)\, \mathrm{d}t.$$

Using the same argument for the continuous function g with $g(x) \leq f(x)$ we arrive at $\lim_{n\to\infty} \frac{1}{n}(f(\alpha) + \ldots + f(n\alpha)) = \int_0^1 f(t)\, \mathrm{d}t$. Therefore,

$$\begin{aligned}\lim_{n\to\infty} \frac{\#\{1 \leq k \leq n:\ \{k\alpha\} \in [a, b]\}}{n} &= \lim_{n\to\infty} \frac{1}{n}\left(f(\alpha) + \ldots + f(n\alpha)\right) \\ &= \int_0^1 f(t)\, \mathrm{d}t = b - a.\end{aligned}$$

□

Problem 5. (10 points) Let $r, b \geq 1$. Prove that there exists a number $R(r, b)$ depending only on r and b with the following property: for every complete graph G with $R(r, b)$ vertices whose edges are coloured red or blue, there exists either a complete subgraph on r vertices which is entirely red, or a complete subgraph on b vertices which is entirely blue.

Solution. This is the so-called Ramsey's theorem. Assume that $R(r, b)$ is the smallest number having the above property. We use induction on $n = r + b$ and prove that

$$R(r, b) \leq R(r-1, b) + R(r, b-1), \quad r, b \geq 1.$$

In the case $r + b = 2$, $r = b = 1$ we trivially have $R(r, b) = 1$. Assume that $R(r-1, b)$ and $R(r, b-1)$ exist and are finite. Take a complete graph V with $R(r-1, b) + R(r, b-1)$ elements and colour its edges. We are to show that there exists a *blue* subgraph of b elements or a *red* subgraph of r element. Take any vertex

$v \in V$. Since $\deg(v) = R(r-1,b) + R(r,b-1) - 1$, there are at least $R(r-1,b)$ red edges incident to v or at least $R(r,b-1)$ blue edges incident to v. Without loss of generality we can assume the first possibility. Consider a subraph of $R(r-1,b)$ vertices adjacent to v. If in this graph there exists a complete *blue* subgraph of b vertices, then trivially our assertion follows. By the induction hypothesis we can therefore assume that there are $r-1$ vertices $v_1, \ldots, v_{r-1}$ that form a *red* subgraph. The graph induced by $v_1, \ldots, v_{r-1}, v$ is *red* and has r vertices. □

Remarks. The above theorem is a cornerstone of the so-called Ramsey theory. The numbers $R(r,b)$ are called Ramsey numbers. The Ramsey numbers $R(k,k)$ are known only for $k \leq 4$. See [R] for more information and open problems on Ramsey numbers.

Problem 6. (15 points)

(a) Let $f : \{0,1,\ldots,n\} \to \{0,1\}$. Prove that there exists the unique polynomial $W : \mathbb{R} \to \mathbb{R}$ with $\deg(W) \leq n$ such that $W(k) = f(k)$ for $0 \leq k \leq n$. Prove that $\deg(W) = 0$ or $\deg(W) \geq n/2$.

(b) Let $f : \{0,1,\ldots,n\} \to \mathbb{R}$ and let us consider the unique polynomial $W : \mathbb{R} \to \mathbb{R}$ with $\deg(W) \leq n$ such that $W(k) = f(k)$ for $0 \leq k \leq n$. Then for $0 \leq r \leq n$ the following are equivalent

   (i) $\deg(W) \leq n - r$,

   (ii) for $n - r < m \leq n$ we have $\sum_{0 \leq j \leq m} (-1)^j \binom{m}{j} f(j) = 0$.

Solution. (a) Let $W(x) = a_n x^n + \ldots + a_0$. The system of equations $W(k) = b_k$, $k = 0,1,\ldots,n$ has always a unique solution due to the fact that $(i^j)_{ij}$ is a Vandermond matrix. To prove the second part we can assume that $\deg(W) > 0$. In the set $\{0,1,\ldots,n\}$ there are at least $n/2$ roots of the polynomial $W(x)$ or at least $n/2$ roots of the polynomial $W(x) - 1$. It follows that $\deg(W) \geq n/2$.

To prove part (b) we first consider the case $r = 1$. The unique interpolation polynomial is given by

$$W(x) = \sum_{j=0}^{n} \left(\prod_{i \neq j} \frac{x-i}{j-i} \right) f(j)$$

The condition $\deg(W) \leq n-1$ is equivalent to the fact that the leading therm in

$W(x)$ vanishes. It suffices to observe that this therm is equal to

$$\sum_{j=0}^{n}\left(\prod_{i\neq j}\frac{1}{j-i}\right)f(j) = \sum_{j=0}^{n}\frac{f(j)}{j!(j-(j+1))\dots(j-n)}$$
$$= \sum_{j=0}^{n}(-1)^{n-j}\frac{f(j)}{j!(n-j)!} = \frac{(-1)^n}{n!}\sum_{j=0}^{n}(-1)^j\binom{n}{j}f(j).$$

We proceed by induction on r. The condition $\deg(W)\leq n-r$ is equivalent to the fact that W is also the unique interpolation polynomial for points $0,1,n-r$ and values $f(0),f(1),\dots,f(n-r)$. The condition $\deg(W)\leq n-r$ is also equivalent to the fact that $\deg(W)\leq n-r+1$ and

$$\sum_{0\leq j\leq m}(-1)^j\binom{m}{j}f(j)=0,\quad m=n-r+1.$$

The inequality $\deg(W)\leq n-r+1$, from the induction hypothesis for $r-1$ and for values $f(0),f(1),\dots,f(n)$ and points $0,1,\dots,n$ is equivalent to

$$\sum_{0\leq j\leq m}(-1)^j\binom{m}{j}f(j)=0,\quad m>n-r+1.$$

$\square$

Remarks. The problem is take from [GR], where the authors suggested the following conjecture.

Open problem 1. Let $f:\{0,1,\dots,n\}\to\{0,1\}$ and consider the unique polynomial $W:\mathbb{R}\to\mathbb{R}$ with $\deg(W)\leq n$ such that $W(k)=f(k)$ for $0\leq k\leq n$. Suppose that f is not constant. Prove that $\deg(W)\geq n-3$. At least, prove that $\deg(W)\geq n-O(1)$.

Problem 7. (5 points) Let $x_1,x_2,\dots,x_n$ be real numbers. Prove the identity

$$\max\{x_1,x_2,\dots,x_n\} = \sum_{i=1}^{n}x_i - \sum_{i<j}\min\{x_i,x_j\} + \sum_{i<j<k}\min\{x_i,x_j,x_k\} - \dots$$
$$+ (-1)^{n+1}\min\{x_1,x_2,\dots,x_n\}.$$

Solution. We can assume that $x_i\geq 0$. Indeed, if it does not hold it suffices to consider $b>0$ such that $x_i+b\geq 0$ for every $i=1,\dots,n$ and to notice that

$$\min\{x_{i_1}+b,\dots,x_{i_k}+b\} = \min\{x_{i_1},\dots,x_{i_k}\}+b.$$

Thus, one also has to verify the identity for $x_1 = x_2 = \cdots = x_n = b$. This case follows from the fact that

$$1 = \binom{n}{1} - \binom{n}{2} + \binom{n}{3} - \cdots + (-1)^n \binom{n}{n},$$

which is equivalent to $0 = (1-1)^n = \sum_{i=0}^n \binom{n}{k}(-1)^k$.

We can also assume that $x_i \in [0,1]$ for every $i = 1, \ldots, n$, since we can always divide these numbers by a sufficiently large constant (the identity is preserved under this operation). To finish the proof it suffices to integrate the identity

$$\mathbf{1}_{[x_1,1]}(t) \cdot \cdots \cdot \mathbf{1}_{[x_n,1]}(t) = (1 - \mathbf{1}_{[0,x_1]}(t)) \cdot \cdots \cdot (1 - \mathbf{1}_{[0,x_n]}(t)).$$

$\square$

Problem 8. (20 points) Let A be a symmetric matrix with eigenvalues $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$.

(a) Prove that for every $k = 1, 2, \ldots, n$ we have

$$\lambda_k = \max_{U:\ \dim(U)=n-k+1} \ \min_{x\in U,\ x\neq 0} \frac{\langle Ax, x\rangle}{\langle x, x\rangle} = \min_{U:\ \dim(U)=k} \ \max_{x\in U,\ x\neq 0} \frac{\langle Ax, x\rangle}{\langle x, x\rangle}.$$

In particular

$$\lambda_1 = \min_{x\neq 0} \frac{\langle Ax, x\rangle}{\langle x, x\rangle}, \qquad \lambda_n = \max_{x\neq 0} \frac{\langle Ax, x\rangle}{\langle x, x\rangle}.$$

(b) Let $A = (a_{ij})_{i,j=1}^n$ be a symmetric matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$. Prove that $\sum_{i,j=1}^n a_{ij}^2 = \sum_{i=1}^n \lambda_i^2$.

(c) We define the operator norm and the Hilbert-Schmidt norm of a real $n \times n$ matrix $A = (a_{ij})$,

$$\|A\| = \sup_{x\in\mathbb{R}^n:\ x\neq 0} \frac{|Ax|}{|x|}, \qquad \|A\|_{HS} = \left(\sum_{ij} a_{ij}^2\right)^{1/2}.$$

Prove that $\|A\|^2$ is the maximal eigenvalue of the matrix A^TA and AA^T. Deduce that is the case of symmetric matrices we have $\|A\| = \max_i |\lambda_i|$. Prove that $\|A\| \le \|A\|_{HS}$.

(d) Let $n \ge 2$ and let $a_{ij} \in \{-1, 1\}$ for $1 \le i < j \le n$. Prove that there exists a vector $x \in \mathbb{R}^n$ with $|x| = 1$ such that $\left|\sum_{1\le i<j\le n}^n a_{ij}x_ix_j\right| \ge c\sqrt{n}$.

Solution. (a) This is the so-called min-max theorem. Let $u_1, \dots, u_n$ be the orthonormal basis for $\mathbb{R}^n$ such that u_i is an eigenvector with an eigenvalue λ_i, $i = 1, \dots, n$. Take a subspace U of $\mathbb{R}^n$ such that $\dim(U) = k$ and take $V = \operatorname{span}\{u_k, \dots, u_n\}$. Note that $U \cap V$ contains a non-zero vector v. Thus, $x = \sum_{i=k}^n a_i u_i$. Therefore,

$$\frac{\langle Ax, x\rangle}{\langle x, x\rangle} = \frac{\sum_{i=k}^n \lambda_i a_i^2}{\sum_{i=k}^n a_i^2} \geq \lambda_k.$$

It follows that

$$\lambda_k \leq \max_{x \in U,\ x \neq 0} \frac{\langle Ax, x\rangle}{\langle x, x\rangle}.$$

and therefore

$$\lambda_k \leq \min_{U:\ \dim(U)=k} \ \max_{x \in U,\ x \neq 0} \frac{\langle Ax, x\rangle}{\langle x, x\rangle}.$$

To see the opposite inequality it suffices to take $U = \operatorname{span}\{u_1, \dots, u_k\}$. Observe that every $x \in U$ has the form $x = \sum_{i=1}^k \lambda_i u_i$.

$$\frac{\langle Ax, x\rangle}{\langle x, x\rangle} = \frac{\sum_{i=1}^k \lambda_i a_i^2}{\sum_{i=1}^k a_i^2} \leq \lambda_k$$

Thus,

$$\max_{x \in U,\ x \neq 0} \frac{\langle Ax, x\rangle}{\langle x, x\rangle} \leq \lambda_k.$$

We get

$$\lambda_k \geq \min_{U:\ \dim(U)=k} \ \max_{x \in U,\ x \neq 0} \frac{\langle Ax, x\rangle}{\langle x, x\rangle}.$$

The equality

$$\lambda_k = \max_{U:\ \dim(U)=n-k+1} \ \min_{x \in U,\ x \neq 0} \frac{\langle Ax, x\rangle}{\langle x, x\rangle}$$

can be proved in a similar way.

(b) This is Problem 1.

(c) Take $x \neq 0$. We have

$$\frac{|Ax|^2}{|x|^2} = \frac{\langle Ax, Ax\rangle}{\langle x, x\rangle} = \frac{\langle A^T Ax, x\rangle}{\langle x, x\rangle}.$$

Thus, the first assertion follows from point (a). If A is symmetric then $A^T A = A^2$ and the eigenvalues of $A^T A$ are $\lambda_1^2, \dots, \lambda_n^2$. Thus,

$$\max_{x \neq 0} \frac{|Ax|^2}{|x|^2} = \max_i \lambda_i^2.$$

Therefore, $\|A\| = \max_i |\lambda_i|$. To prove that $\|A\|^2$ is also equal to the maximal eigenvalue of AA^T is suffices to prove that the spectrum of AA^T and the spectrum of A^TA are equal. We present the solution due to Kapitan Orlef teem. Take a sequence $e_1, e_2, \ldots$ converging to 0 such that $A_\varepsilon = A + \varepsilon I$ is invertible. This sequence exists since $\det(A + \varepsilon I) = 0$ is a polynomial equation in ε and therefore has only finite solutions. Let $\varepsilon = \varepsilon_n$. We have

$$\det(A_\varepsilon B - tI) = \det(A_\varepsilon)\det(B - tA_\varepsilon^{-1}) = \det(B - tA_\varepsilon^{-1})\det(A_\varepsilon) = \det(BA_\varepsilon - tI).$$

Taking $\varepsilon = \varepsilon_n \to 0$ we get $\det(AB - tI) = \det(BA - tI)$. Thus, the spectrum of AB is the same as the spectrum of BA.

Let $0 \le \mu_1 \le \mu_2 \le \cdots \le \mu_n$ be eigenvalues of a symmetric matrix A^TA. We have

$$\|A\| = (\max_i \mu_i)^{1/2} \le \left(\sum_i \mu_i\right)^{1/2} = \sqrt{\operatorname{tr}(A^TA)} = \|A\|_{HS}.$$

(d) Define the matrix $A = (A_{ij})$ as follows,

$$A_{ij} = \begin{cases} a_{ij}/2 & i < j \\ a_{ji}/2 & i > j \\ 0 & i = j \end{cases}.$$

The matrix A is symmetric and $|A_{ij}| = 1/2$ for $i \neq j$. Let $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$ be the eigenvalues of A. From point (a) we have

$$\max_{|x|=1}\left|\sum_{1\le i<j\le n} a_{ij}x_ix_j\right| = \max_{|x|=1}|\langle Ax, x\rangle| = \max_i |\lambda_i| \ge \left(\frac{\sum_{i=1}^n \lambda_i^2}{n}\right)^{1/2} = n^{-1/2}\|A\|_{HS}$$

$$= n^{-1/2}\left((n^2 - n)/4\right)^{1/2} \ge \frac{\sqrt{2}}{4}\sqrt{n}.$$

□

Problem 9. (10 points) We say that a polygon P (a subset of a plane bounded by a piecewise linear curve without self-intersections) has an ear at a vertex V if the line V_-V_+, where V_-, V_+ are adjacent to V lies entirely inside the polygon P. Two ears are said to be non-overlapping if the interiors of triangles VV_-V_+ are disjoint.

(a) Prove that except for triangles, every polygon has at least two non-overlapping ears.

(b) Prove that there exists a triangulation of P with no additional vertices and a 3-coloring of the vertices of P such that each triangle does not have two vertices with the same colour.

(c) The art gallery has a shape of an polygon P with n-vertices. Show that one can place $[n/3]$ guards in vertices of P who together can observe the whole gallery.

Proof. (a) We provide a sketch of the proof, for details see [M]. The assertion is clearly true for quadrilaterals. We proceed by induction of the number of vertices of our polygon. Suppose P is an polygon with $n > 4$ vertices. Select a vertex V of P at which the interior angle is less than $180°$, and let V_- and V_+ denote the vertices of P which are adjacent to V.

We consider the case when VV_-V_+ is an ear. Let us call it E_0. If we remove this ear, then the remaining polygon P' has $n-1$ vertices and therefore it has at least two ears E_1, E_2. One of pairs (E_0, E_1), (E_0, E_2) must be a pair of non-overlapping ears.

Suppose that VV_-V_+ is not an ear. Then the triangle VV_-V_+ must contain a vertex in the interior or on the chord V_-V_+. Let Z be such a vertex with an additional property that the line through it and parallel to V_-V_+ is as close to V as possible. Hence the chord VZ lies entirely inside the polygon P and so divides it into two polygons. Each of them has at least two ears, say E_1^1, E_2^1 for the first polygon and E_1^2, E_2^2 for the second one. Only two of them can have VZ as an edge. Therefore the remaining two are non-overlapping ears of P.

(b) We proceed by induction. The assertion for triangles is trivial. Take a polygon with n-vertices. From point (a) we know that P has an ear $E = V_0V_1V_2$ at a vertex V_0. We can remove this ear by removing V_0 and obtain a polygon P' with $n-1$ vertices. From the induction assumption we know that P' admits a triangulation with a good 3-coloring. Now it suffices to color the removed vertex V_0 with a colour different than the colours of V_1 and V_2.

(c) This proof is due to [F]. Take a coloring from point (b) with colours a, b, c. Let V_a, V_b, V_c be the sets of vertices having colours a, b, c, respectively. We can assume that $|V_a| \leq |V_b| \leq |V_c|$. Then $|V_a| \leq [n/3]$. It is now immediate to see that if we place guards in vertices from the set V_a then they together can observe the whole gallery (since they observe each triangle). □

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Selected theorems in mathematics

Part III, prepared by: Piotr Nayar

Problem 1. (20 points) Let $x_1, x_2, \ldots, x_n$ be vectors in a Euclidean space $(\mathbb{R}^N, \|\cdot\|)$ and let $2 \le k \le n$. Prove the inequality

$$\binom{n-2}{k-2}\left(\sum_{i=1}^n \|x_i\| - \left\|\sum_{i=1}^n x_i\right\|\right) \le \sum_{1\le i_1<i_2<\cdots<i_k\le n} \left(\|x_{i_1}\| + \cdots + \|x_{i_k}\| - \|x_{i_1} + \cdots + x_{i_k}\|\right).$$

In particular, prove that if x, y, z are vectors in a Euclidean space $(\mathbb{R}^N, \|\cdot\|)$ then we have

$$\|x+y\| + \|y+z\| + \|z+x\| \le \|x\| + \|y\| + \|z\| + \|x+y+z\|.$$

Proof. This inequality is due to Djoković, see [D]. By a straightforward computation we prove the following identity,

$$\binom{n-2}{k-2}\left(\left(\sum_{i=1}^n \|x_i\|\right)^2 - \left\|\sum_{i=1}^n x_i\right\|^2\right) = \sum_{1\le i_1<i_2<\cdots<i_k\le n} \left((\|x_{i_1}\| + \cdots + \|x_{i_k}\|)^2 - \|x_{i_1} + \cdots + x_{i_k}\|^2\right).$$

This is the Adamović identity, see [A]. Our inequality follows from this identity and the inequality

$$\|x_{i_1}\| + \cdots + \|x_{i_k}\| + \|x_{i_1} + \cdots + x_{i_k}\| \le \sum_{i=1}^n \|x_i\| + \left\|\sum_{i=1}^n x_i\right\|,$$

which is equivalent to the triangle inequality,

$$\left\|\sum_{i\in I} x_i\right\| \le \left\|\sum_{i\notin I} x_i\right\| + \left\|\sum_{i=1}^n x_i\right\| \le \sum_{i\notin I}\|x_i\| + \left\|\sum_{i=1}^n x_i\right\|,$$

where $I = \{i_1, \ldots, i_k\}$.

The inequality

$$\|x+y\| + \|y+z\| + \|z+x\| \le \|x\| + \|y\| + \|z\| + \|x+y+z\|$$

is the famous Hlawka's inequality. This inequality follows immediately by taking $k = 2$ and $n = 3$ in the Djoković inequality. $\square$

Problem 2. (20 points) Let $z_1, z_2, \dots, z_n$ be complex numbers. Prove that there exists a subset I of $\{1, 2, \dots, n\}$ such that

$$\left|\sum_{k\in I} z_k\right| \ge \frac{1}{\pi}\sum_{k=1}^{n} |z_k|.$$

Is the constant $1/\pi$ optimal?

Solution. For a real number x we write $x^+ = \max\{x, 0\}$. Let $z_k = r_k(\cos\theta_k + i\sin\theta_k)$, where $r_k = |z_k|$ and $\theta_k \in [0, 2\pi)$, $k = 1, \dots, n$. We define

$$f(\theta) = \sum_{k=1}^{n} r_k(\cos(\theta - \theta_k))^+.$$

We have

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi} f(\theta)\,\mathrm{d}\theta &= \frac{1}{2\pi}\sum_{k=1}^{n} r_k \int_0^{2\pi} (\cos(\theta-\theta_k))^+\,\mathrm{d}\theta = \frac{1}{2\pi}\sum_{k=1}^{n} r_k \int_{-\pi/2}^{\pi/2}\cos(\theta)\,\mathrm{d}\theta \\ &= \frac{1}{\pi}\sum_{k=1}^{n} r_k = \frac{1}{\pi}\sum_{k=1}^{n}|z_k|.\end{aligned}$$

Thus, there exists $\theta \in [0, 2\pi)$ such that $f(\theta) \ge \frac{1}{\pi}\sum_{k=1}^{n}|z_k|$. We fix this θ. Let

$$I = \{1 \le k \le n : \cos(\theta - \theta_k) > 0\}.$$

Then,

$$\begin{aligned}\left|\sum_{k\in I} z_k\right| &= \left|e^{-i\theta}\sum_{k\in I} z_k\right| = \left|\sum_{k\in I} r_k e^{i(\theta_k-\theta)}\right| \ge \mathrm{Re}\left(\sum_{k\in I} r_k e^{i(\theta_k-\theta)}\right) \\ &= \sum_{k\in I} r_k\cos(\theta_k-\theta) = \sum_{k\in I} r_k\cos(\theta-\theta_k) = \sum_{k=1}^{n} r_k(\cos(\theta-\theta_k))^+ \\ &= f(\theta) \ge \frac{1}{\pi}\sum_{k=1}^{n}|z_k|.\end{aligned}$$

The constant $1/\pi$ is optimal. To see this we take $z_k = \exp(2(k-1)\pi i/n)$, $k = 1, \ldots, n$. Let I be a subset of $\{1, \ldots, n\}$ such that $|\sum_{k \in I} z_k|$ is maximal and let $v = \sum_{k \in I} z_k$. It is easy to see that

$$\{1 \leq k \leq n : \langle v, z_k \rangle \geq 0\} \subseteq I.$$

Indeed, if $\langle v, z_k \rangle > 0$ and $k \notin I$ then $|v + z_k| \geq |v|$, which contradicts the definition of I. Similarly, we have

$$\{1 \leq k \leq n : \langle v, z_k \rangle < 0\} \subseteq \{1, \ldots, n\} \backslash I.$$

Indeed, if $\langle v, z_k \rangle < 0$ and $k \in I$ then $|v - z_k| \geq |v|$, so we can remove k from I and increase the value of $|\sum_{k \in I} z_k|$.

In particular, we can assume that $I = \{1, \ldots, m\}$ for some $m \in \{1, \ldots, n\}$. In this case we have

$$\begin{aligned} \frac{\left|\sum_{k \in I} z_k\right|}{\sum_{k=1}^n |z_k|} &= \frac{1}{n} \left| \sum_{k=0}^{m-1} e^{\frac{2\pi i k}{n}} \right| = \frac{1}{n} \left| \frac{e^{\frac{2\pi i m}{n}} - 1}{e^{\frac{2\pi i}{n}} - 1} \right| \leq \frac{2}{n} \left| \frac{1}{e^{\frac{2\pi i}{n}} - 1} \right| \\ &= \frac{2}{n} \cdot \frac{1}{|e^{\frac{\pi i}{n}} - e^{\frac{-\pi i}{n}}|} = \frac{1}{n|\sin \frac{\pi}{n}|}. \end{aligned}$$

Thus, for every $I \subseteq \{1, \ldots, n\}$ we have

$$\left| \sum_{k \in I} z_k \right| \leq \frac{1}{n \sin \frac{\pi}{n}} \sum_{k=1}^n |z_k|.$$

Taking $n \to \infty$ we obtain $\lim_{n \to \infty} \frac{1}{n \sin \frac{\pi}{n}} = 1/\pi$. $\square$

Problem 3. (20 points) Consider a $n \times m$ matrix A with $0, 1$ entries. We assume that the number of 1's in the matrix A equals $2j$, where j is an integer. Is it always possible to remove some number of columns and rows of A is such a way that the number of 1's in the remaining matrix is j?

Solution. The solution is due to Prof. Keith Ball. The answer is no. It suffices to consider the following 5×9 matrix with 44 entries equal to 1,

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

It is easy to see that the number of 1's in the matrix A, after removing some number of rows a and columns, must be equal to kl or $kl-1$ for some integers $0 \le k \le 5$ and $0 \le l \le 9$. On the other hand it must be equal to 22. In this range the equations $kl = 22 = 2 \cdot 11$ and $kl = 23$ do not have a solution. □

Problem 4. (30 points)

(a) Let A and B be non-empty compact sets in $\mathbb{R}$. Prove that for every $\lambda \in [0,1]$ we have
$$|\lambda A + (1-\lambda)B| \ge (1-\lambda)|A| + \lambda|B|.$$

(b) Let f, g and m be nonnegative measurable functions on $\mathbb{R}$ and let $\lambda \in [0,1]$. Assume that for all $x, y \in \mathbb{R}$ we have
$$m((1-\lambda)x + \lambda y) \ge f(x)^{1-\lambda} g(y)^{\lambda}.$$
Prove that
$$\int_{\mathbb{R}} m \ge \left(\int_{\mathbb{R}} f\right)^{1-\lambda} \left(\int_{\mathbb{R}} g\right)^{\lambda}. \tag{1}$$

(c) Prove the inequality (1) in $\mathbb{R}^n$.

Solution. (a) The proofs are taken from [GNT], where one can find historical remarks on the statements. The inequality from point (a) is the one-dimensional Brunn-Minkowski inequality. Observe that the operations $A \to A + v_1$, $B \to B + v_2$ where $v_1, v_2 \in \mathbb{R}$ do not change the volumes of A, B and $(1-\lambda)A + \lambda B$ (adding a number to one of the sets only shifts all of this sets). Therefore we can assume that $\sup A = \inf B = 0$. But then, since $0 \in A$ and $0 \in B$, we have
$$(1-\lambda)A + \lambda B \supset (1-\lambda)A \cup (\lambda B).$$
But $(1-\lambda)A$ and (λB) are disjoint, up to the one point 0. Therefore
$$|(1-\lambda)A + \lambda B| \ge |(1-\lambda)A| + |\lambda B|.$$

(b) This is the Prékopa-Leindler inequality in dimension 1. We can assume, considering $f\mathbf{1}_{f \le M}$ and $g\mathbf{1}_{g \le M}$ instead of f and g, that f, g are bounded. Note also that this inequality possesses some homogeneity. Indeed, if we multiply f, g, m by numbers c_f, c_g, c_m satisfying
$$c_m = c_f^{1-\lambda} c_g^{\lambda},$$

then the hypothesis and the assertion do not change. Therefore, taking $c_f = \|f\|_\infty^{-1}$, $c_g = \|g\|_\infty^{-1}$ and $c_m = \|f\|_\infty^{-(1-\lambda)} \|g\|_\infty^{-\lambda}$ we can assume (since we are in the situation when f and g are bounded) that $\|f\|_\infty = \|g\|_\infty = 1$. But then

$$\int_{\mathbb{R}} m = \int_0^{+\infty} |\{m \geq s\}| \, \mathrm{d}s,$$

$$\int_{\mathbb{R}} f = \int_0^1 |\{f \geq r\}| \, \mathrm{d}r,$$

$$\int_{\mathbb{R}} g = \int_0^1 |\{g \geq r\}| \, \mathrm{d}r.$$

Note also that if $x \in \{f \geq r\}$ and $y \in \{g \geq r\}$ then by the assumption of the theorem we have $(1-\lambda)x + \lambda y \in \{m \geq r\}$. Hence,

$$(1-\lambda)\{f \geq r\} + \lambda\{g \geq r\} \subset \{m \geq r\}.$$

Moreover, the sets $\{f \geq r\}$ and $\{g \geq r\}$ are non-empty for $r \in [0,1)$. This is very important since we want to use the 1-dimensional Brunn-Minkowski inequality proved in step (a). For any non empty compact subsets $A \subset \{f \geq r\}$ and $B \subset \{g \geq r\}$ we have $|\{m \geq r\}| \geq (1-\lambda)|A| + \lambda|B|$. Since Lebesgue measure is inner regular, we get that

$$|\{m \geq r\}| \geq (1-\lambda)|\{f \geq r\}| + \lambda|\{g \geq r\}|.$$

We have

$$\begin{aligned}
\int m = \int_0^{+\infty} |\{m \geq r\}| \, \mathrm{d}r \geq \int_0^1 |\{m \geq r\}| \, \mathrm{d}r \geq \int_0^1 |(1-\lambda)\{f \geq r\} + \lambda\{g \geq r\}| \, \mathrm{d}r \\
\geq (1-\lambda) \int_0^1 |\{f \geq r\}| \, \mathrm{d}r + \lambda \int_0^1 |\{g \geq r\}| \, \mathrm{d}r = (1-\lambda) \int f + \lambda \int g \\
\geq \left(\int f\right)^{1-\lambda} \left(\int g\right)^{\lambda}.
\end{aligned}$$

Observe that we have proved

$$\int m \geq (1-\lambda) \int f + \lambda \int g,$$

but this inequality does not have the previous homogeneity, hence it requires the assumption $\|f\|_\infty = \|g\|_\infty = 1$ to hold.

(c) (the inductive step). Suppose our inequality is true in dimension $n-1$. We will prove it in dimension n.

Suppose we have numbers $y_0, y_1, y_2 \in \mathbb{R}$ satisfying $y_0 = (1-\lambda)y_1 + \lambda y_2$. Define $m_{y_0}, f_{y_1}, g_{y_2} : \mathbb{R}^{n-1} \to \mathbb{R}_+$ by

$$m_{y_0}(x) = m(y_0, x), \quad f_{y_1}(x) = f(y_1, x), \quad g_{y_2}(x) = (y_2, x),$$

where $x \in \mathbb{R}^{n-1}$. Note that since $y_0 = (1-\lambda)y_1 + \lambda y_2$ we have

$$\begin{aligned} m_{y_0}((1-\lambda)x_1 + \lambda x_2) &= m((1-\lambda)y_1 + \lambda y_2, (1-\lambda)x_1 + \lambda x_2) \\ &\geq f(y_1, x_1)^{1-\lambda} g(y_2, x_2)^{\lambda} = f_{y_1}(x_1)^{1-\lambda} g_{y_2}(x_2)^{\lambda}, \end{aligned}$$

hence m_{y_0}, f_{y_1} and g_{y_2} satisfy the assumption of the $(n-1)$-dimensional Prékopa-Leindler inequality. Therefore we have

$$\int_{\mathbb{R}^{n-1}} m_{y_0} \geq \left(\int_{\mathbb{R}^{n-1}} f_{y_1} \right)^{1-\lambda} \left(\int_{\mathbb{R}^{n-1}} g_{y_2} \right)^{\lambda}.$$

Define new functions $M, F, G : \mathbb{R} \to \mathbb{R}_+$

$$M(y_0) = \int_{\mathbb{R}^{n-1}} m_{y_0}, \quad F(y_1) = \int_{\mathbb{R}^{n-1}} f_{y_1}, \quad G(y_2) = \int_{\mathbb{R}^{n-1}} g_{y_2}.$$

We have seen (the above inequality) that when $y_0 = (1-\lambda)y_1 + \lambda y_2$ then there holds

$$M((1-\lambda)y_1 + \lambda y_2) \geq F(y_1)^{1-\lambda} G(y_2)^{\lambda}.$$

Hence, by 1-dimensional Prékopa-Leindler inequality proved in Step 1, we get

$$\int_{\mathbb{R}} M \geq \left(\int_{\mathbb{R}} F \right)^{1-\lambda} \left(\int_{\mathbb{R}} G \right)^{\lambda}.$$

But

$$\int_{\mathbb{R}} M = \int_{\mathbb{R}^n} m, \quad \int_{\mathbb{R}} F = \int_{\mathbb{R}^n} f, \quad \int_{\mathbb{R}} G = \int_{\mathbb{R}^n} g,$$

so we conclude that

$$\int_{\mathbb{R}^n} m \geq \left(\int_{\mathbb{R}^n} f \right)^{1-\lambda} \left(\int_{\mathbb{R}^n} g \right)^{\lambda}.$$

□

Problem 5. (15 points)

(a) Prove that any sequence of real numbers $x_1, x_2, \ldots$ contains a non-increasing or a non-decreasing subsequence.

(b) Let $n, m \geq 1$ be integers. Suppose we have a sequence of $(n-1)(m-1)+1$ real numbers. Prove that there exists a non-decreasing sequence of length n or a non-increasing sequence of length m.

Solution. (a) The assertion clearly holds when $x_1, x_2, \ldots$ is not bounded (take a monotone sequence converging to ∞ or to $-\infty$). If our sequence is bounded then from the Bolzano-Weierstrass theorem we can find its converging subsequence $A = \{x_{i_1}, x_{i_2}, \ldots\}$. Let g be the limit of this subsequence. One of the sets $A \cap (-\infty, g]$, $A \cap [g, \infty)$ is infinite. In the first case we can find a non-decreasing subsequence of A and in the second case we can find a non-increasing subsequence of A.

(b) This is the Erdös-Szekeres theorem, see [ES]. The presented proof can be found in [AZ]. Assume, by way of contradiction, that there is no non-decreasing sequence of length n. Define the function $f : \{1, 2, \ldots, (n-1)(m-1)+1\} \to \{1, 2, \ldots, n-1\}$ in the following way,

$$f(i) = \text{length of the longest increasing subsequence that ends with } x_i.$$

The function f has domain of size $(n-1)(m-1)+1$ and the range of size $n-1$. Thus, there exist $i_1 < i_2 < \cdots < i_m$ and a number $k \in \{1, \ldots, n-1\}$ such that

$$f(x_{i_1}) = f(x_{i_2}) = \cdots = f(x_{i_m}) = k.$$

Note that $x_{i_j} > x_{i_{j+1}}$ since otherwise $f(x_{i_{j+1}}) = k+1$ (add the point $x_{i_{j+1}}$ to the longest sequence that ends with x_{i_j}). Thus, the sequence

$$x_{i_1} > x_{i_2} > \cdots > x_{i_m}$$

is a decreasing sequence of length m. $\square$

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Selected theorems in mathematics

Part IV, prepared by: Piotr Nayar

Problem 1. (15 points) Let $A = (a_{ij})_{i,j=1}^n$ be a skew-symmetric real matrix, i.e. $A^T = -A$. Prove that there exists a polynomial P in variables a_{ij} such that $\det(A) = P^2$.

Solution. We present the proof published in [P]. Note that if n is odd then

$$\det(A) = \det(-A^T) = (-1)^n \det(A^T) = (-1)^n \det(A) = -\det(A).$$

Thus, $\det(A) = 0$. It suffices to consider the case when n is even and $\det(A) \neq 0$. We proceed by the induction on n. In the case $n = 2$ we clearly have $a_{11} = a_{22} = 0$ and $a_{12} = -a_{21}$. Thus $\det(A) = a_{12}^2$.

Take $n \geq 4$. Let M_{ij} be the (i,j)-th minor of A (i.e. the determinant of a matrix obtained by removing the ith row and jth column of A). Let $A_{ij} = (-1)^{i+j} M_{ij}$. We have

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} A_{11} & A_{21} & \dots & A_{n1} \\ A_{12} & A_{22} & \dots & A_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ A_{1n} & A_{2n} & \dots & A_{nn} \end{bmatrix}. \tag{1}$$

Let $\Delta_n = \det(a_{ij})_{i,j=1}^n$ and $\Delta_{n-2} = \det(a_{ij})_{i,j=3}^n$. From (1) we have

$$\det \begin{bmatrix} A_{11} & A_{21} & \dots & A_{n1} \\ A_{12} & A_{22} & \dots & A_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ A_{1n} & A_{2n} & \dots & A_{nn} \end{bmatrix} = \Delta_n^{n-1}.$$

Moreover, we have

$$\begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{bmatrix} \times \begin{bmatrix} A_{11} & A_{21} & \dots & A_{n1} \\ A_{12} & A_{22} & \dots & A_{n2} \\ A_{13} & A_{23} & \dots & A_{n3} \\ \vdots & \vdots & \ddots & \vdots \\ A_{1n} & A_{2n} & \dots & A_{nn} \end{bmatrix}$$
$$= \begin{bmatrix} A_{11} & A_{21} & A_{31} & \dots & A_{n1} \\ A_{12} & A_{22} & A_{32} & \dots & A_{n2} \\ 0 & 0 & \Delta_n & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \Delta_n \end{bmatrix}.$$

Computing the determinant of both sides gives

$$\Delta_{n-2}\Delta_n^{n-1} = \Delta_n^{n-2} \cdot \det \begin{bmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{bmatrix}.$$

Since A_{11} and A_{22} are determinants of skew symmetric matrices of odd size we obtain $A_{11} = A_{22} = 0$. Moreover, $M_{12}^T = -M_{21}$ and therefore $A_{12} = -A_{21}$. Thus, $\Delta_{n-2}\Delta_n = -A_{21}A_{12} = A_{12}^2$. By the induction assumption we know that Δ_{n-2} is a square of some polynomial. Since the ring of all multivariate polynomial over a field is a unique factorization domain, we get that Δ_n must also be a square of a certain polynomial. $\square$

Problem 2. (15 points)

(a) Prove the Brunn-Minkowski inequality, which states that if A and B are non-empty compact sets then for all $\lambda \in [0, 1]$ we have

$$|(1-\lambda)A + \lambda B| \geq |A|^{1-\lambda}|B|^{\lambda}$$

and

$$|(1-\lambda)A + \lambda B|^{1/n} \geq (1-\lambda)|A|^{1/n} + \lambda|B|^{1/n}.$$

(b) Prove the isoperimetric inequality, i.e., show that when $|A| = |B|$, where A is a measurable set in $\mathbb{R}^n$ and B is an Euclidean ball in $\mathbb{R}^n$, then $|A_t| \geq |B_t|$, where $A_t = \{x \in \mathbb{R}^n,\ \mathrm{dist}(x, A) \leq t\}$.

(c) Let A be a compact subset of $\mathbb{R}^n$ and let us define

$$|\partial A| = \liminf_{t\to 0^+} \frac{|A + tB_2^n| - |A|}{t},$$

where B_2^n is an Euclidean ball. Show that the condition $|A| = |B|$, where B is a Euclidean ball in $\mathbb{R}^n$ implies $|\partial A| \geq |\partial B|$.

Solution. (a) To prove the first statement it suffices to use Prékopa-Leindler inequality (Part III, Problem 4) for function $f = \mathbf{1}_A$, $g = \mathbf{1}_B$ and $m = \mathbf{1}_{(1-\lambda)A+\lambda B}$. To deduce the second inequality we take

$$\mu = \frac{\lambda |B|^{1/n}}{(1-\lambda)|A|^{1/n} + \lambda |B|^{1/n}}.$$

Then

$$\begin{aligned}\left|\frac{(1-\lambda)A + \lambda B}{(1-\lambda)(\operatorname{vol} A)^{1/n} + \lambda(\operatorname{vol} B)^{1/n}}\right| &= \left|(1-\mu)\frac{A}{(\operatorname{vol} A)^{1/n}} + \mu \frac{B}{(\operatorname{vol} B)^{1/n}}\right| \\ &\geq \left|\frac{A}{|A|^{1/n}}\right|^{1-\mu} \left|\frac{B}{|B|^{1/n}}\right|^{\mu} = 1.\end{aligned}$$

(b) The Brunn-Minkowski inequality yields the isoperimetric inequality for the Lebesgue measure on $\mathbb{R}^n$. Indeed, suppose we have a compact set $A \subset \mathbb{R}^n$ and let B be a Euclidean ball of the radius r_A such that $|B| = |A|$. Then from the Brunn-Minkowski inequality we have

$$\begin{aligned}|A_t|^{1/n} &= |A + tB_2^n|^{1/n} \geq |A|^{1/n} + |tB_2^n|^{1/n} \\ &= |B_2^n|^{1/n} r_A + |B_2^n|^{1/n} t = |B + tB_2^n|^{1/n} = |B_t|^{1/n}.\end{aligned}$$

It means that

$$|A_t| \geq (r_A + t)^n |B_2^n| = |B_t|.$$

(c) We have

$$|\partial A| = \liminf_{t\to 0^+} \frac{|A + tB_2^n| - |A|}{t} = \liminf_{t\to 0^+} \frac{|A_t| - |A|}{t} \geq \liminf_{t\to 0^+} \frac{|B_t| - |B|}{t},$$

and therefore $|\partial A| \geq |\partial B|$. One can also deduce that

$$|\partial A| \geq n r_a^{n-1} |B_2^n| = n \left(\frac{|A|}{|B_2^n|}\right)^{\frac{n-1}{n}} = n |B_2^n|^{1/n} |A|^{\frac{n-1}{n}}.$$

$\square$

Problem 3. (10 points) Fix $1 \le k \le n$. Let $A_1, A_2, \ldots, A_m$ be distinct subsets of $\{1, 2, \ldots, n\}$ such that $|A_i \cap A_j| = k$ for all $i \neq j$. Prove that $m \le n$.

Solution. This is the so-called Fisher's inequality. Consider a matrix $A = (a_{ij})_{i=1,j=1}^{m,n}$ where $a_{ij} = |A_i \cap A_j|$. Let $v_1, \ldots, v_m \in \mathbb{R}^n$ be the rows of A. It suffices to prove that these vectors are linear independent. Suppose, by contradiction, that for some $\lambda_1, \ldots, \lambda_m$ we have $\sum_{i=1}^m \lambda_i v_i = 0$, with not all coefficients being zero. Note that $\langle v_i, v_j \rangle = k$ for $i \neq j$ and $\langle v_i, v_i \rangle = |A_i|$ for $i = 1, \ldots, m$. We have

$$\begin{aligned} 0 = \left\langle \sum_{i=1}^m \lambda_i v_i, \sum_{i=1}^m \lambda_i v_i \right\rangle &= \sum_{i=1}^m \lambda_i^2 |A_i| + k \sum_{i \neq j} \lambda_i \lambda_j \\ &= \sum_{i=1}^m \lambda_i^2 (|A_i| - k) + k \left(\sum_{i=1}^m \lambda_i \right)^2. \end{aligned}$$

It follos that $|A_1| = |A_2| = \ldots = |A_m| = k$. This contradicts the condition $|A_1 \cap A_2| = k$ and $A_1 \neq A_2$. □

Problem 4. (10 points) Suppose that G is a graph on n vertices with more than $n^2/4$ edges. Prove that G contains a triangle. Show that for an even number n there exists a graph G with n vertices and $n^2/4$ edges containing no triangle.

Solution. Assume that G has no triangles. Let m be the number of edges in G and let V be the set of vertices. Let $\{x, y\}$ be an edge of G. The vertices x, y have no common neighbours. Thus, $d(x) + d(y) \le n$. We obtain

$$\sum_{x \in V} d(x)^2 = \sum_{\{x,y\} \in E} (d(x) + d(y)) \le mn.$$

On the other hand, by the Cauchy-Schwarz inequality we have

$$\sum_{x \in V} d(x)^2 \ge \frac{1}{|V|} \left(\sum_{x \in V} d(x) \right)^2 = \frac{4m^2}{n}.$$

Thus, $m \le n^2/4$, a contradiction.

To give an example of a graph on n vertices (n even) containing $n^2/4$ edges and no triangle it suffices to consider the complete $\frac{n}{2} \times \frac{n}{2}$ bipartite graph.

□

Problem 5. (20 points) Let X, Y be independent identically distributed real random variables. Prove that

$$\mathbb{E}|X + Y| \ge \mathbb{E}|X - Y|.$$

Solution. This inequality comes from the paper [B]. Define the function $f : \mathbb{R}^2 \to \mathbb{R}$,

$$f(s,x) = \operatorname{sgn}(x)\mathbf{1}_{\{s:|s|\le|x|\}}(s).$$

It is easy to verify that

$$\int_{\mathbb{R}} f(s,x)f(s,y)\,\mathrm{d}s = |x+y| - |x-y|.$$

Indeed, for $x, y \ge 0$ both sides are clearly equal to $\min\{x,y\}$. To get the other cases it suffices to observe that both sides are invariant under changing the signs of x and y.

We arrive at

$$\begin{aligned}\mathbb{E}\left(|X+Y| - |X-Y|\right) &= \mathbb{E}\int_{\mathbb{R}} f(s,X)f(s,Y)\,\mathrm{d}s = \int_{\mathbb{R}} \mathbb{E}(f(s,X)f(s,Y))\,\mathrm{d}s \\ &= \int_{\mathbb{R}} \mathbb{E}f(s,X)\mathbb{E}f(s,Y)\,\mathrm{d}s = \int_{\mathbb{R}} (\mathbb{E}f(s,X))^2\,\mathrm{d}s \ge 0,\end{aligned}$$

where in the second inequality we have used Fubini's theorem, the third equality follows from the fact that X, Y are independent, and the fourth equality – from the fact that they have the same distribution. □

Problem 6. (20 points) Let $(\mathbb{Z}^d, E)$ be an integer lattice, i.e., a graph such that $\{x,y\} \in E$ if and only if $|x-y| = 1$. A path from x_0 to x_n is a sequence $x_0, x_1, \ldots, x_n \in \mathbb{Z}^d$ such that $\{x_i, x_{i+1}\} \in E$ for $i = 0, 1, \ldots, n-1$. Such a path is called a path of length n from x_0 to x_n. For $u, v \in \mathbb{Z}^d$ let $P^k(u,v)$ be the number of paths from u to v having length k. Prove that for every $k \ge 1$ and every $u, v \in \mathbb{Z}^d$ we have $P^{2k}(u,u) \ge P^{2k}(u,v)$.

Solution. Note that for every sequence of real numbers $a_1, a_2 \ldots, a_n$ and every permutation $\pi : [n] \to [n]$ we have $\sum_{i=1}^n a_i^2 \ge \sum_{i=1}^n a_i a_{\pi(i)}$. Indeed, we have

$$\sum_{i=1}^n a_i^2 - \sum_{i=1}^n a_i a_{\pi(i)} = \frac{1}{2}\sum_{i=1}^n (a_i - a_{\pi(i)})^2 \ge 0.$$

Clearly, for every $u, w \in \mathbb{Z}^d$ we have $P^k(u,w) = P^k(w,u)$. Thus,

$$P^{2k}(u,v) = \sum_{w\in\mathbb{Z}^d} P^k(u,w)P^k(w,v) = \sum_{w\in\mathbb{Z}^d} P^k(u,w)P^k(v,w).$$

The two vectors $(P^k(u,w))_{w\in\mathbb{Z}^d}$ and $(P^k(v,w))_{w\in\mathbb{Z}^d}$ can be obtained from each other by permuting the coordinates. Thus, from the above fact the right hand side in the above equality is maximal for $u = v$. □

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Selected theorems in mathematics

Part VI, prepared by: Piotr Nayar

Problem 1. (20 points) Let C be a smooth closed curve on the unit sphere S^2 of length less then 2π. Prove that this curve is contained in a certain open hemisphere.

Solution. Consider two points P, Q on our curve that divide it into two curves C_1, C_2 of the same length. Then the distance from P to Q along the sphere is less than π so there is a unique minor arc from P to Q. Let M be the midpoint of this arc. We show that no point of G hits the equatorial great circle with M as north pole. Suppose, by contradiction, that C_1 hits the equator at a point A. Then we may construct a curve $\tilde{C}_1$ by rotating C_1 one-half turn about the axis through M. Clearly in this procedure P goes to Q, Q goes to P while A goes to the antipodal point $\tilde{A}$. The curve $C_1 \cup \tilde{C}_1$ has the same length as C and contains two antipodal points $A, \tilde{A}$. Thus, the length of this curve is greater or equal 2π. This is a contradiction. $\square$

Problem 2. (10 points) Let A be a measurable set on S^1 with $|A| = \pi$. Prove that there exists a complex number z with $|z| = 1$ such that $|A \cup (zA)| \geq \frac{3}{2}\pi$.

Solution. We identify S^1 with an interval $[0, 2\pi)$. Let A be a set in $[0, 2\pi)$ with $|A| = \pi$. Let us consider a quantity $|A \cap ((A + t) \bmod 2\pi)|$, where $t \in [0, 2\pi)$. Note that usin Fubini's theorem we have

$$\frac{1}{2\pi}\int_0^{2\pi} |A \cap ((A+t) \bmod 2\pi)| \,\mathrm{d}t = \frac{1}{2\pi}\int_0^{2\pi}\int_0^{2\pi} \mathbf{1}_A(s)\mathbf{1}_A((s-t) \bmod 2\pi) \,\mathrm{d}s\,\mathrm{d}t$$
$$= \frac{1}{2\pi}\int_0^{2\pi}\int_0^{2\pi} \mathbf{1}_A(s)\mathbf{1}_A((s-t) \bmod 2\pi) \,\mathrm{d}t\,\mathrm{d}s = \frac{1}{2\pi}\int_0^{2\pi} \mathbf{1}_A(s)|A| \,\mathrm{d}s = \frac{1}{2\pi}|A|^2 = \frac{\pi}{2}.$$

Thus, there exists t_0 such that $|A \cap ((A + t_0) \bmod 2\pi)| \leq \pi/2$. It follows that

$$|A \cup ((A+t_0) \bmod 2\pi)| \geq |A| + |(A+t_0) \bmod 2\pi| - |A \cap ((A+t_0) \bmod 2\pi)| \geq \pi + \pi - \pi/2 = \frac{3}{2}\pi.$$

$\square$

Problem 3. (20 points) Let $u_1, u_2, \ldots, u_m$ be non-zero vectors in the Euclidean space $\mathbb{R}^n$ satisfying the condition $\langle u_i, u_j \rangle \leq 0$ for all $i \neq j$.

(a) Prove that if $\sum_{i \in I} \alpha_i u_i = 0$ for some $I \subset [m]$ and non-zero $\alpha_i, i \in I$ and for every $J \subset I$, $J \neq I$ we have $\sum_{i \in J} \alpha_i u_i \neq 0$ then all the numbers α_i have the same sign.

(b) Prove that $m \leq 2n$.

(c) Let $d \geq 1$. Prove that $C_1, \ldots, C_m$ are binary vectors of length $2d$ such that for all $i \neq j$ vectors C_i and C_j have different signs on at least d coordinates then $m \leq 4d$.

Solution. The solution is taken from [W].

(a) Suppose that $\sum_{i\in I}\alpha_i u_i = 0$. We can assume that for every $J \subset I$, $J \neq I$ we have $\sum_{i\in J}\alpha_i u_i \neq 0$. Indeed, if $\alpha_{i_1}u_{i_1} + \alpha_{i_2}u_{i_2} = 0$ and $\alpha_{i_1} < 0 < \alpha_{i_2}$ then

$$0 = \alpha_{i_1}\langle u_{i_1}, u_{i_1}\rangle + \alpha_{i_2}\langle u_{i_1}, u_{i_2}\rangle < 0,$$

which is a contradiction. Therefore $\alpha_{i_1}u_{i_1} + \alpha_{i_2}u_{i_2} \neq 0$ and we can construct a minimal set I such that $\{i_1, i_2\} \subset I$.

Suppose that not all $\alpha_i, i \in I$ have the same sign. Then we have a partition $I = I_1 \cup I_2$ such that $\sum_{i\in I_1}\beta_i u_i = \sum_{i\in I_2}\beta_i u_i$ and $\beta_i > 0$ for all $i \in I$. Let $w = \sum_{i\in I_1}\beta_i u_i$. We have

$$0 \leq \langle w, w\rangle = \left\langle \sum_{i\in I_1}\beta_i u_i, \sum_{i\in I_2}\beta_i u_i \right\rangle = \sum_{i\in I_1, j\in I_2}\beta_i\beta_j\langle u_i, u_j\rangle \leq 0.$$

It follows that $w = 0$. This contradicts the minimality of I.

(b) We proceed by the induction on n. The case $n = 1$ is trivial. We can assume that $m > n \geq 2$. Let $\{w_1, \ldots, w_r\}$ be a minimal subset of linearly independent vectors chosen from $\{u_1, \ldots, u_m\}$. We write

$$\{u_1, \ldots, u_m\} = \{w_1, \ldots, w_r\} \cup \{w_1, \ldots, w_{m-r}\}.$$

Choose $\alpha_1, \ldots, \alpha_r$ such that $\sum_{i=1}^r \alpha_i w_i = 0$. By the first part we can assume that $\alpha_j > 0$. For each v_j we have

$$0 = \langle v_j, w\rangle = \sum_{i=1}^r \alpha_i\langle w_i, v_j\rangle \leq 0,$$

therefore we must have $\langle w_i, v_j\rangle = 0$. Thus, the subspaces spanned by $(w_i)_i$ and $(v_j)_j$ are orthogonal. The space spanned by $(w_i)_i$ has dimension $r-1$ so $(v_j)j$ lie in a subspace of dimension $n-r+1$. By the induction assumption we have $m - r \leq 2(n-r+1)$. Thus, $m \leq 2n - r + 2 \leq 2n$.

(c) if we represent $C_1, \ldots, C_m$ as elements of $\{-1, 1\}^{2d}$ then $\langle C_i, C_j\rangle \leq 0$ for $i \neq j$. From (b) we deduce that $m \leq 4d$. This bound in called the Plotkin bound. □

Problem 4. (15 points) Let $f(x) = \sum_{k=n}^m a_n \sin(kx)$. Prove that f has at least $2n$ zeros in the interval $[0, 2\pi)$.

Solution. We can assume that $a_n > 0$. Take a function

$$f_l(x) = \sum_{k=n}^m \frac{a_k}{k^{2l}}(-1)^l \sin(kx).$$

Clearly, for sufficiently large l we have

$$\left|\frac{a_n}{n^{2l}}\right| > \sum_{k>n}\left|\frac{a_k}{k^{2l}}\right|.$$

In this case

$$f_l\left(\frac{2k\pi}{n} - \frac{3\pi}{2n}\right) > 0, \qquad f_l\left(\frac{2k\pi}{n} - \frac{\pi}{2n}\right) < 0, \qquad k = 1, 2, \ldots, n.$$

From the mean value property there exist point $x_1, x_2, \ldots, x_{2n} \in [0, 2\pi)$ such that $f_l(x_k) = 0$ for $k - 1, \ldots, 2n$. Using Role's theorem $2l$ times we deduce that $f = \frac{\mathrm{d}^{2l}}{\mathrm{d}x^{2l}} f_l$ also has at least $2n$ zeros. □

Problem 5. (15 points) Let $2k \leq n$ and let $\mathcal{A}$ be a family of subsets of $[n]$ such that each subset in of size k and for every $A, B \in \mathcal{A}$ we have $A \cap B \neq \emptyset$. Prove that $|\mathcal{A}| \leq \binom{n-1}{k-1}$.

Solution. This is the so-called Erdös-Ko-Rado theorem. The idea is to count pairs (π, S) where π is a circular permutation $(\pi(1), \pi(2), \ldots, \pi(n))$ and S is an interval of length k in this permutation such that $S \in \mathcal{A}$. In other words S is an interval on the discrete circle, where the numbers are placed according to π and the elements in this interval must form a set from $\mathcal{A}$. We have $(n-1)!$ cyclic permutations. Each of them contains at most k pairwise intersecting intervals of length k and thus at most k elements of our family. In this step we have used the fact that $2k \leq n$. Each set in our family occurs in precisely $k!(n-k)!$ cyclic permutations. Thus,

$$|\mathcal{A}|k!(n-k)! \leq k(n-1)!.$$

Our assertion follows. □

References

[W] Wildon's Weblog, http://wildonblog.wordpress.com/2011/01/