FKN Theorem on the biased cube

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Abstract

In this note we consider Boolean functions defined on the discrete cube $\{-\gamma, \gamma^{-1}\}^n$ equipped with a product probability measure $\mu^{\otimes n}$, where $\mu = \beta\delta_{-\gamma} + \alpha\delta_{\gamma^{-1}}$ and $\gamma = \sqrt{\alpha/\beta}$. This normalization ensures that the coordinate functions $(x_i)_{i=1,\ldots,n}$ are orthonormal in $L_2(\{-\gamma, \gamma^{-1}\}^n, \mu^{\otimes n})$. We prove that if the spectrum of a Boolean function is concentrated on the first two Fourier levels, then the function is close to a certain function of one variable. Our theorem strengthens the non-symmetric FKN theorem due to Jendrej, Oleszkiewicz and Wojtaszczyk.

Moreover, in the symmetric case $\alpha = \beta = \frac{1}{2}$ we prove that if a $[-1,1]$-valued function defined on the discrete cube is close to a certain affine function, then it is also close to a $[-1,1]$-valued affine function.

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1 Introduction and notation

Let $\alpha, \beta > 0$ with $\alpha + \beta = 1$ and $\alpha \in (0, \frac{1}{2})$. We consider the discrete cube $\{-\gamma, \gamma^{-1}\}^n$ equipped with the L_2 structure given by the product probability measure $\mu_n = \mu^{\otimes n}$, where $\mu = \beta\delta_{-\gamma} + \alpha\delta_{\gamma^{-1}}$ and $\gamma = \sqrt{\alpha/\beta}$. For $f, g : \{-\gamma, \gamma^{-1}\}^n \to \mathbb{R}$ let us define the expectation $\mathbb{E}f = \int f \, \mathrm{d}\mu_n$, the standard scalar product $\langle f, g \rangle = \mathbb{E}fg$ and the induced norm $\|f\| = \sqrt{\langle f, f \rangle}$. We also

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define the L_p norm, $\|f\|_p = (\mathbb{E}|f|^p)^{1/p}$. Let $[n] = \{1, 2 \ldots, n\}$. For $T \subseteq [n]$ and $x = (x_1, \ldots, x_n)$ let $w_T(x) = \prod_{i \in T} x_i$ and $w_\emptyset \equiv 1$. Note that we have $\mathbb{E}x_i = 0$ and $\mathbb{E}x_i x_j = \delta_{ij}$. It follows that $(w_T)_{T \subseteq [n]}$ is an orthonormal basis of $L_2(\{-\gamma, \gamma^{-1}\}^n, \mu_n)$. Therefore, every function $f : \{-\gamma, \gamma^{-1}\}^n \to \mathbb{R}$ admits the unique expansion $f = \sum_{T \subseteq [n]} a_T w_T$. The functions w_T are sometimes called the Walsh-Fourier functions. If the function f is $\{-1, 1\}$-valued then it is called Boolean.

The Fourier analysis of Boolean functions plays an important role in many areas of research, including learning theory, social choice, complexity theory and random graphs, see e.g. [O1] and [O2]. One of the most important analytic tools in this theory is the so-called hypercontractive Bonami-Beckner-Gross inequality, see [Bo], [Be], [G1] and [G2] for a survey on this topic. This inequality has been used in the celebrated papers by J. Kahn, G. Kalai and N. Linial, [KKL], and E. Friedgut, [F]. It can be stated as follows. Take $\alpha = \beta = \frac{1}{2}$ and $q \in [1, 2]$. Then we have

$$\left\| \sum_{T \subseteq [n]} (q-1)^{|T|/2} a_T w_T \right\|_2 \leq \left\| \sum_{T \subseteq [n]} a_T w_T \right\|_q \tag{1}$$

for every choice of $a_T \in \mathbb{R}$. This inequality has been generalized in [Ol1] to the non-symmetric case. Namely, the following inequality holds true,

$$\left\| \sum_{T \subseteq [n]} c_q(\alpha, \beta)^{|T|} a_T w_T \right\|_2 \leq \left\| \sum_{T \subseteq [n]} a_T w_T \right\|_q, \tag{2}$$

where

$$c_q(\alpha, \beta) = \sqrt{\frac{\beta^{2-\frac{2}{q}} - \alpha^{2-\frac{2}{q}}}{\alpha\beta\left(\alpha^{-\frac{2}{q}} - \beta^{-\frac{2}{q}}\right)}}.$$

One can easily check that (1) is a special case of (2), namely $\sqrt{q-1} = \lim_{\varepsilon \to 0} c_q(\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon)$. Moreover, it is easy to see that $c_q(\alpha, \beta) \in [0, 1]$.

In [FKN] the authors proved the following theorem, which is now called the FKN Theorem. Suppose $\alpha = \beta = \frac{1}{2}$ and we have a Boolean function f whose Fourier spectrum is concentrated on the first two levels, say $\sum_{|T|>1} a_T^2 < \varepsilon^2$. Then f is $C\varepsilon$-close in the L_2 norm to the constant function or to one of the functions $\pm x_i$. Here and in what follows C is a universal constant that may vary from one line to another. The authors gave two proofs of this theorem. One of them contained an omission which was fixed by G. Kindler and S. Safra in their unpublished paper, [KS], see also [K].

The FKN Theorem was originally devised for applications in discrete combinatorics and social choice theory. It is useful in the proof of the robust version of the famous Arrow's theorem on Condorcet's voting paradox, see [A] and [KG]. It was also applied is theoretical computer science, e.g., it is useful in analyzing the Long Code Test in the proof of the PCP theorem by I. Dinur, [D]. Also the FKN Theorem in the biased case is worthy of attention, e.g., p-biased long code was used by I. Dinur and S. Safra in their PCP proof of NP-hardness of approximation of the Vertex Cover problem, see [DS].

In [JOW] the authors gave a proof of the following version of the FKN Theorem,

Theorem 1 ([JOW], Theorem 5.3 and Theorem 5.8)**.** *Let $f = \sum_T a_T w_T$ be the Walsh-Fourier expansion of a function $f : \{-1,1\}^n \to \{-1,1\}$ and let $\rho = \left(\sum_{|T|>1} a_T^2\right)^{1/2}$. Then there exists $B \subseteq [n]$ with $|B| \le 1$ such that $\sum_{|T|\le 1, T\neq B} a_T^2 \le C\rho^4 \ln(2/\rho)$ and $|a_B|^2 \ge 1-\rho^2 - C\rho^4\ln(2/\rho)$. In particular,*

$$\mathrm{dist}_{L_2}(f, w_B) \le \rho + C\rho^2 \ln(2/\rho). \tag{3}$$

Moreover, in the non-symmetric case, $f : \{-\gamma,\gamma^{-1}\}^n \to \{-1,1\}$, there exists $k \in [n]$ such that $\left\| f - (a_\emptyset + a_{\{k\}} w_{\{k\}})\right\| \le 8\sqrt{\rho}$.

The inequalities (3) is sharp, up to the universal constants. In the proof the inequality (1) has been used. However, in the non-symmetric case one can ask for a better bound involving bias parameter α. In this note we use inequality (2) to prove such an extension of the FKN Theorem. Namely, we have

Theorem 2. *Let $f = \sum_T a_T w_T$ be the Walsh-Fourier expansion of a function $f : \{-\gamma,\gamma^{-1}\}^n \to \{-1,1\}$ and let $\rho = \left(\sum_{|T|>1} a_T^2\right)^{1/2}$. Then there exists $k \in [n]$ such that for $\rho \ln(e^2/\rho) < \frac{3}{2^{10}e^4}\alpha$ we have*

$$\left\| f - (a_\emptyset + a_{\{k\}} w_{\{k\}})\right\| \le 2\rho \tag{4}$$

and

$$\left\| f - sgn(a_\emptyset + a_{\{k\}} w_{\{k\}})\right\| \le 4\rho. \tag{5}$$

In this paper we use the $\{-1,1\}$-valued function $\mathrm{sgn}(x) = -\mathbb{I}_{(-\infty,0)}(x) + \mathbb{I}_{[0,\infty)}(x)$.

Our proof of Theorem 2, which is given in the Section 2, is an application of the ideas used in the proof of Theorem 5.3 in [JOW]. Our inequality is closely related to the inequality of A. Rubinstein, see [R, Corollary 10]. Rubinstein's inequality states that for every function $f : \{-\gamma, \gamma^{-1}\}^n \to \{-1,1\}$ with $\sum_{|T|>1} a_T^2 = \rho^2$ we have

$$\left\| f - (a_\emptyset + a_{\{k\}} w_{\{k\}}) \right\| \le \frac{K\rho}{(1-a_\emptyset^2)^{1/2}}, \quad K = 13104. \tag{6}$$

However, our inequality (4) is a better bound in the regime $\rho \ln(e/\rho) < c_0 \alpha$. To see this consider the case when $f_0 = \mathrm{sgn}(a_\emptyset + a_{\{k\}} w_{\{k\}})$ is constant and equal to $\varepsilon \in \{-1,1\}$. Then from (5) we have $\|f - \varepsilon\|^2 \le 16\rho^2$. It follows that $1 - a_\emptyset^2 = \|f - \mathbb{E}f\|^2 \le \|f - \varepsilon\|^2 \le 16\rho^2$. Thus, the right hand side of (6) is greater than $K/4$, which gives no information. In the case when f_0 is not constant we have $|\mathbb{E}f_0| = |1 - 2\alpha|$. Thus,

$$||a_\emptyset| - |1-2\alpha|| = ||\mathbb{E}f| - |\mathbb{E}f_0|| \le |\mathbb{E}(f - f_0)| \le \|f - f_0\| \le 4\rho.$$

It follows that $1 - a_\emptyset^2 \le 2(1 - |a_\emptyset|) \le 2(2\alpha + 4\rho) \le 12\alpha$. Therefore, the right hand side in the Rubinstein bound is in this case $K\rho/\sqrt{12\alpha}$ which is much greater than ρ when $\alpha \to 0$.

In the Section 3 we consider the case $\gamma = 1$ and we deal with the problem concerning $[-1,1]$-valued functions defined on the cube $\{-1,1\}^n$ with uniform product probability measure. A function $f : \{-1,1\}^n \to \mathbb{R}$ is called *affine* if $f(x) = a_0 + \sum_{i=1}^n a_i x_i$, where $a_0, a_1, \ldots, a_n \in \mathbb{R}$ and $x = (x_1, \ldots, x_n)$. We will denote the set of all affine functions by $\mathcal{A}$. Moreover, let $\mathcal{A}_{[-1,1]} \subseteq \mathcal{A}$ stands for the set of all affine functions satisfying $|f(x)| \le 1$ for every $x \in \{-1,1\}^n$. Note that $f \in \mathcal{A}_{[-1,1]}$ if and only if $\sum_{i=0}^n |a_i| \le 1$. The function $f(x) = x_i$ will be denoted by r_i, $i = 1, \ldots, n$. Let us also notice that if f is $[-1,1]$-valued then $|a_T| = |\mathbb{E}w_T f| \le \mathbb{E}|w_T f| \le 1$.

In [JOW] the authors gave the following example. Take $g : \{-1,1\}^n \to \mathbb{R}$ given by $g(x) = s^{-1} n^{-1/2} \sum_{i=1}^n x_i$. Note that $g \in \mathcal{A}$. Define $\phi(x) = -\mathbf{1}_{(-\infty,-1)}(x) + x\mathbf{1}_{[-1,1]}(x) + \mathbf{1}_{(1,\infty)}(x)$ and take $f = \phi \circ g$. Clearly, f is $[-1,1]$-valued but may not be affine. The authors proved that $\lim_{n\to\infty} \mathrm{dist}_{L^2}(f, \mathcal{A}) = O(e^{-s^2/4})$ and $\lim_{n\to\infty} \mathrm{dist}_{L^2}(f, \mathcal{A}_{[-1,1]}) = \Theta(s^{-1})$.

Here we prove that this is the worst case as far as the dependence of these two quantities is concerned. Namely, we have the following theorem, which is the analogue of (3) in the case of $[-1,1]$-valued functions.

Theorem 3. *Let us take* $f : \{-1,1\}^n \to [-1,1]$ *and define* $\rho = \mathrm{dist}_{L^2}(f, \mathcal{A})$. *Then* $\mathrm{dist}_{L_2}(f, \mathcal{A}_{[-1,1]}) \le \frac{18}{\sqrt{\ln(1/\rho)}}$.

2 Proof of Theorem 2

In this section we give a proof of Theorem 2. We begin with a simple lemma.

Lemma 1. *Let* $0 < \alpha < \beta < 1$ *with* $\alpha + \beta = 1$ *and let* $\gamma \in (0,1]$. *Then*

$$\frac{\alpha^{-2+\gamma} - \beta^{-2+\gamma}}{\beta^\gamma - \alpha^\gamma} \le \frac{2-\gamma}{\gamma} \cdot \frac{\alpha^{-2+\gamma}}{\beta^\gamma}.$$

Proof. Let $x \in (0,1)$ and $\mu \ge 1$. From the mean value theorem we have $\frac{1-x^\mu}{1-x} \le \mu$. Applying this with $\mu = \frac{2-\gamma}{\gamma}$ and $x = (\alpha/\beta)^\gamma$ yields an equivalent version of the statement. □

Proof of Theorem 2. Let k be given by Theorem 1. Let $h = f - (a_\emptyset + a_{\{k\}} x_k)$ and $\tilde{h} = f - \mathrm{sgn}(a_\emptyset + a_{\{k\}} x_k)$. Moreover, let $\delta = \|h\|$. It follows that $\delta \le 1$. Note that for every $u \in \mathbb{R}$ and $\varepsilon \in \{-1,1\}$ we have $|u - \mathrm{sgn}(u)| \le |u - \varepsilon|$. Therefore,

$$|\varepsilon - \mathrm{sgn}(u)| \le |\varepsilon - u| + |u - \mathrm{sgn}(u)| \le 2|u - \varepsilon|. \tag{7}$$

It follows that $|\tilde{h}| \le 2|h|$. Thus, using the fact that $\tilde{h}$ is $\{-2,0,2\}$-valued, we have

$$\mathbb{P}(\tilde{h} \ne 0) = \frac{1}{4}\|\tilde{h}\|^2 \le \|h\|^2 = \delta^2.$$

Let us consider the expansion $\tilde{h} = \sum_T \tilde{a}_T w_T$. Clearly, $\tilde{a}_T = a_T$ for $T \ne \emptyset, \{k\}$. Using (2) we obtain

$$\begin{aligned} 4\delta^{4/q} \ge 4\mathbb{P}(\tilde{h} \ne 0)^{2/q} = \|\tilde{h}\|_q^2 = \left\| \sum_T \tilde{a}_T w_T \right\|_q^2 \ge \left\| \sum_T c_q(\alpha,\beta)^{|T|} \tilde{a}_T w_T \right\|_2^2 \\ = \sum_T c_q(\alpha,\beta)^{2|T|} \tilde{a}_T^2 \ge c_q(\alpha,\beta)^2 \sum_{|T| \le 1} \tilde{a}_T^2, \end{aligned}$$

where $q \in [1,2]$. Using Lemma 1 with $\gamma = 2 - 2/q$ we obtain

$$\sum_{|T|\le 1,\ T \ne \emptyset, \{k\}} \tilde{a}_T^2 \le \sum_{|T| \le 1} \tilde{a}_T^2 \le \frac{4\delta^{4/q}}{c_q(\alpha,\beta)^2} = 4\delta^{4/q}\alpha\beta \cdot \frac{\alpha^{-\frac{2}{q}} - \beta^{-\frac{2}{q}}}{\beta^{2-\frac{2}{q}} - \alpha^{2-\frac{2}{q}}} \le \frac{4\delta^{4/q}}{q-1}\left(\frac{\alpha}{\beta}\right)^{1-\frac{2}{q}}.$$

Take $\frac{1}{q} = 1 - \frac{1}{\ln(e^2/\delta)} \in [\frac{1}{2}, 1]$. Note that $(\alpha/\beta)^{1-2/q} \le \alpha^{1-2/q} \le \alpha^{-1}$. It follows that

$$\sum_{|T|\le 1,\ T\ne\emptyset,\{k\}} \tilde{a}_T^2 \le 4\delta^4\alpha^{-1}e^{\frac{4\ln(1/\delta)}{\ln(e^2/\delta)}} \ln\left(e^2/\delta\right) \le 4e^4\delta^4\alpha^{-1}\ln\left(e^2/\delta\right).$$

From Theorem 1 we have $\rho \le \delta \le 8\sqrt{\rho}$. Thus,

$$4e^4\delta^4\alpha^{-1}\ln\left(e^2/\delta\right) \le 2^8e^4\alpha^{-1}\delta^2\rho\ln(e^2/\rho) \le \frac{3}{4}\delta^2.$$

Note that $a_\emptyset^2 + a_{\{k\}}^2 = 1 - \delta^2$. We deduce

$$1 - \rho^2 = \sum_{|T|\le 1} a_T^2 = a_\emptyset^2 + a_{\{k\}}^2 + \sum_{|T|\le 1,\ T\ne\emptyset,\{k\}} \tilde{a}_T^2 \le 1 - \delta^2 + \frac{3}{4}\delta^2 = 1 - \frac{1}{4}\delta^2.$$

Therefore, $\delta \le 2\rho$.

The inequality (5) follows from (7). □

Remark. The condition $\rho\ln(e^2/\rho) \le \frac{1}{2^9e^4}\alpha$ cannot be significantly improved. Indeed, if we take $f : \{-\gamma, \gamma^{-1}\}^2 \to \{-1, 1\}$ given by

$$f(x_1, x_2) = 2(\beta - \sqrt{\beta\alpha}x_1)(\beta - \sqrt{\beta\alpha}x_2) - 1,$$

see the remark after the proof of Theorem 5.8 in [JOW], then we obtain $\rho = 2\alpha\beta \le 2\alpha$ and $\delta = 2\beta^{3/2}\alpha^{1/2}$. Thus $\delta = \sqrt{2\rho}\beta \ge \sqrt{\rho/2}$.

One can easily see that if we replace our assumption $\rho\ln(e^2/\rho) \le \frac{1}{2^9e^4}\alpha$ by a slightly stronger condition, say $\rho\ln^2(e^2/\rho) \le \alpha$ then we obtain $\delta \le \rho + o(\rho)$, which means that $\sum_{|T|\le 1,\ T\ne\emptyset,\{k\}} a_T^2 = o(\rho^2)$ and $a_\emptyset^2 + a_{\{k\}}^2 \ge 1 - \rho^2 - o(\rho^2)$.

3 Proof of Theorem 3

We need the following lemma due to P. Hitczenko, S. Kwapień and K. Oleszkiewicz.

Lemma 2. *([HK], Theorem 1 and [Ol2], Theorem 1) Let $a_1 \ge a_2 \ge \ldots \ge a_n \ge 0$ and let us take $S : \{-1, 1\}^n \to \mathbb{R}$ given by $S = \sum_{i=1}^n a_ir_i$. Then for $t \ge 1$ we have*

$$\mathbb{P}\left(|S| \ge \|S\|\right) > \frac{1}{10} \tag{8}$$

and

$$\|S\|_t \geq \frac{1}{4}\sqrt{t}\Big(\sum_{i>t} a_i^2\Big)^{1/2}. \tag{9}$$

We give a proof of Theorem 3.

Proof of Theorem 3 . Step 1. If $f = \sum_T a_T w_T$ then $\mathrm{dist}_{L_2}(f, \mathcal{A}) = \|f - S\|$, where $S = \sum_{|T|\leq 1} a_T w_T$. For every $u \in [-1,1]$ we have $|x-u| \geq |x-\phi(x)|$ for all $x \in \mathbb{R}$. Taking $x = S$ and $u = f$ we obtain $\mathbb{E}(|S|-1)_+^2 = \|S - \phi(S)\|^2 \leq \|S - f\|^2 \leq \rho^2$. For all $g \in \mathcal{A}_{[-1,1]}$ we have

$$\|g - f\| \leq \|g - S\| + \|S - f\| \leq \|g - S\| + \rho.$$

Therefore,

$$\mathrm{dist}_{L_2}(f, \mathcal{A}_{[-1,1]}) \leq \mathrm{dist}_{L_2}(S, \mathcal{A}_{[-1,1]}) + \rho. \tag{10}$$

It suffices to prove that $\mathbb{E}(|S|-1)_+^2 \leq \rho^2$ implies an appropriate bound on $\mathrm{dist}_{L_2}(S, \mathcal{A}_{[-1,1]})$, whenever $S = a_0 + \sum_{i=1}^n a_i r_i$, where $a_0, a_1, \ldots, a_n \in \mathbb{R}$.

Step 2. Suppose that for all $n \geq 1$ we can prove that $\mathbb{E}(|S|-1)_+^2 \leq \rho^2$ implies $\mathrm{dist}_{L_2}(S, \mathcal{A}_{[-1,1]}) \leq M$ for some $M > 0$, assuming that $a_0 = 0$. Then we can deal with the case $a_0 \neq 0$ as follows. Let us take $\tilde{S} : \{-1,1\} \times \{-1,1\}^n \to \mathbb{R}$ given by $\tilde{S} = a_0 x_0 + \sum_{i=1}^n a_i x_i$. Clearly, $\mathbb{E}(|\tilde{S}|-1)_+^2 = \mathbb{E}(|S|-1)_+^2 \leq \rho^2$. We can find a $[-1,1]$-valued function $\tilde{S}_0 = b_0 x_0 + \sum_{i=1}^n b_i x_i$ such that $\left\|\tilde{S} - \tilde{S}_0\right\| \leq M$. Take $S_0 = b_0 + \sum_{i=1}^n b_i x_i$. Now it suffices to observe that the function S_0 is $[-1,1]$-valued and to notice that $\left\|\tilde{S} - \tilde{S}_0\right\| = \|S - S_0\|$.

Step 3. Take $S = \sum_{i=1}^n a_i r_i$. Without loss of generality we can assume that $1 \geq a_1 \geq a_2 \geq \ldots \geq a_n \geq 0$. Let $\tau = \max\{t \geq 1 : \sum_{i=1}^t a_i \leq 1\}$. Clearly, $\tau \geq 1$. If f is already in $\mathcal{A}_{[-1,1]}$ then there is nothing to prove. Therefore we can assume that $\tau < n$. We can also assume that $\rho \leq 1/3$, since otherwise we have

$$\mathrm{dist}_{L_2}(f, \mathcal{A}_{[-1,1]}) \leq \mathrm{dist}_{L_2}(f, 0) = \|f\| \leq 1 \leq \frac{18}{\sqrt{\ln(1/\rho)}}.$$

Let $A = \{|S| \geq \frac{1}{2}\|S\|_t\}$. For $t \geq 1$ we have

$$\mathbb{E}|S|^t = \mathbb{E}|S|^t \mathbf{1}_A + \mathbb{E}|S|^t \mathbf{1}_{A^c} \leq \sqrt{\mathbb{E}|S|^{2t}}\sqrt{\mathbb{P}(A)} + \frac{1}{2^t}\mathbb{E}|S|^t.$$

Since by the Khinchine inequality we have $(\mathbb{E}|S|^{2t})^{1/2t} \le \sqrt{\frac{2t-1}{t-1}}(\mathbb{E}|S|^t)^{1/t}$, we arrive at

$$\mathbb{P}\left(|S| \ge \frac{1}{2}\left\|S\right\|_t\right) \ge \left(1-\frac{1}{2^t}\right)^2 \frac{(\mathbb{E}|S|^t)^2}{\mathbb{E}|S|^{2t}} \ge \frac{1}{4}\frac{(\mathbb{E}|S|^t)^2}{\mathbb{E}|S|^{2t}} \ge \frac{1}{4}\left(\frac{t-1}{2t-1}\right)^t.$$

By the Chebyshev inequality we obtain

$$\mathbb{P}\left(|S| \ge 1+\varepsilon\right) \le \frac{\mathbb{E}(|S|-1)_+^2}{\varepsilon^2} \le \frac{\rho^2}{\varepsilon^2}, \tag{11}$$

for all $\varepsilon > 0$. Let $t \ge 1$ and assume that $\left\|S\right\|_t > 2$. Take $\varepsilon = \frac{1}{2}\left\|S\right\|_t - 1 > 0$. We get

$$\frac{1}{4}\left(\frac{t-1}{2t-1}\right)^t \le \mathbb{P}\left(|S| \ge \frac{1}{2}\left\|S\right\|_t\right) \le \frac{\rho^2}{\left(\frac{1}{2}\left\|S\right\|_t - 1\right)^2}.$$

It follows that

$$\left\|S\right\|_t \le 2 + 4\rho\left(\frac{2t-1}{t-1}\right)^{t/2}$$

which is also true in the case $\left\|S\right\|_t \le 2$. From inequality (9) we obtain

$$\frac{1}{4}\sqrt{t}\Big(\sum_{i>t} a_i^2\Big)^{1/2} \le \left\|S\right\|_t \le 2 + 4\rho\left(\frac{2t-1}{t-1}\right)^{t/2}. \tag{12}$$

Step 4. We consider the case $\tau \ge \frac{2}{\ln 3}\ln(1/\rho) \ge 1$. Let us now take $t = \frac{2}{\ln 3}\ln(1/\rho) \ge 2 > 1$ and define $S_1 = \sum_{i \le \frac{2}{\ln 3}\ln(1/\rho)} a_i r_i$. Notice that $\sum_{i \le \frac{2}{\ln 3}\ln(1/\rho)} a_i \le \sum_{i\le\tau} a_i \le 1$. Thus, $S_1 \in \mathcal{A}_{[-1,1]}$. Moreover, since $t \ge 2$, we have $\rho\left(\frac{2t-1}{t-1}\right)^{t/2} \le \rho 3^{t/2} = 1$ and therefore by (12) we have

$$\mathrm{dist}_{L_2}(S, \mathcal{A}_{[-1,1]}) \le \|S - S_1\| = \left(\sum_{i>\frac{2}{\ln 3}\ln(1/\rho)} a_i^2\right)^{1/2} \le \frac{24}{\sqrt{\frac{2}{\ln 3}\ln(1/\rho)}}.$$

In this case (10) yields

$$\mathrm{dist}_{L_2}(f, \mathcal{A}_{[-1,1]}) \le \frac{24}{\sqrt{\frac{2}{\ln 3}\ln(1/\rho)}} + \rho \le \frac{18}{\sqrt{\ln(1/\rho)}}.$$

Step 5. We are to deal with the case $\tau < \frac{2}{\ln 3}\ln(1/\rho)$. Let us take $S_2 = \sum_{i\geq\tau+2} a_i r_i$. From inequality (8) we have

$$\mathbb{P}\left(|S| \geq \sum_{i\leq\tau+1} a_i + \|S_2\|\right) \geq \frac{1}{2^{\tau+1}}\mathbb{P}\left(|S_2| \geq \|S_2\|\right) \geq \frac{1}{2^{\tau+1}} \cdot \frac{1}{10} \geq \frac{1}{20}\rho^{\frac{2\ln 2}{\ln 3}}.$$

Note that $\sum_{i\leq\tau+1} a_i > 1$. Therefore, from inequality (11) we obtain

$$\mathbb{P}\left(|S| \geq \sum_{i\leq\tau+1} a_i + \|S_2\|\right) \leq \frac{\rho^2}{\left(\sum_{i\leq\tau+1} a_i + \|S_2\| - 1\right)^2}.$$

It follows that

$$\sum_{i\leq\tau+1} a_i + \|S_2\| - 1 \leq \sqrt{20}\rho^{1-\frac{\ln 2}{\ln 3}}.$$

Take $S_1 = \sum_{i=1}^{\tau} a_i r_i + (1-(a_1+\ldots+a_\tau))r_{\tau+1}$. Clearly, $S_1 \in \mathcal{A}_{[-1,1]}$. Moreover,

$$\begin{aligned}\|S - S_1\| &= \left((1-(a_1+\ldots+a_\tau) - a_{\tau+1})^2 + \|S_2\|^2\right)^{1/2} \\ &\leq |a_1+\ldots+a_\tau+a_{\tau+1}-1| + \|S_2\| \leq \sqrt{20}\rho^{1-\frac{\ln 2}{\ln 3}}.\end{aligned}$$

Therefore, from (10) we have

$$\mathrm{dist}_{L_2}(f, \mathcal{A}_{[-1,1]}) \leq \sqrt{20}\rho^{1-\frac{\ln 2}{\ln 3}} + \rho \leq \frac{18}{\sqrt{\ln(1/\rho)}}.$$

□

Remark. If we perform our calculation with $\ln(2.03)$ instead of $\ln 3$ we will obtain the theorem with a constant 14.5 instead of 18.

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