Global weak solutions to a sixth order Cahn-Hilliard type equation

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Abstract

In this paper we study a sixth order Cahn-Hilliard type equation that arises as a model for the faceting of a growing surface. We show global in time existence of weak solutions and uniform in time a priori estimates in the H^3 norm. These bounds enable us to show the uniqueness of weak solutions.

Key words: Cahn-Hilliard equation, self-assembly, global weak solution

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1 Introduction

During the last two or three decades it has become popular to model the evolution of thin solid films in terms of continuum theory. One example for a thin film approximation of a surface diffusion based process that describes the faceting of a growing surface has been given by Savina et al. [3]. It can be extended to more complex self-assembly systems such as quantum dots [5-8]. However here, we stick to the one-material model established before. Additional information on self-arranging nano-surfaces, quantum dots and faceting of growing surfaces can be found in the references mentioned above. Mathematically, the problem is interesting and challenging, since the regularizing Wilmore term in the surface energy results, when applying a long wave approximation, in a sixth order term that dominates the semilinear partial differential equation. More precisely, the model describes an evolving surface, a graph of function $h : \Omega \subset \mathbb{R}^2 \times [0, T] \to \mathbb{R}$. The surface is governed in Ω by

$$h_t = \frac{D}{2}|\nabla h|^2 + \Delta^2 h + \Delta^3 h - \Delta[\beta(h_y^2 h_{xx} + h_x^2 h_{yy} + 4h_x h_y h_{xy}) + \alpha(h_x^2 h_{xx} + h_y^2 h_{yy})]. \quad (1)$$

Here, $\alpha, \beta > 0$ are anisotropy coefficients, $D > 0$ is a parameter related to the deposition rate, Δ is the standard Laplacian and subscripts indicate differentiation with respect to the noted variables. Furthermore, as described in the derivation of this equation (see Savina et al. [3] or Korzec [6]), the overall surface is in a moving frame. As usually, an initial condition supplements the problem,

$$h(x, y, 0) = h_0(x, y), \quad \text{for} \quad (x, y) \in \Omega \tag{2}$$

and also boundary conditions have to be imposed. There are various possibilities, but the two most common ones are given by defining the domain as

$$\Omega = \mathbb{R}^2 \quad \text{or} \quad \Omega = \mathbb{T}^2 \quad ,$$

where $\mathbb{T}^2$ is the flat torus. The latter one yields a periodic surface, it seems as realistic as an infinite domain. Hence we choose the bounded version to gain additional technical advantages in the analysis.

We establish the existence of global weak solutions, i.e. we show that there exists a function $h \in C([0,T], H^3)$ with $h_t \in L_\infty((0,T), H^{-3})$, such that h satisfies (1) in the distributional sense.

The main result is stated below, it will be proved in Section 3.

Theorem 1 *Let us assume that $h_0 \in H^3(\mathbb{T}^2)$, then there exists a unique weak solution (1), which is well-defined on $[0, \infty)$.*

Before we proceed with the proof, we want to record the structure of the problem, which has also been found in the originating paper [3]. Basically, equation (1) is a perturbed gradient system

$$h_t = \frac{D}{2}|\nabla h|^2 + \Delta \mathcal{H}. \tag{3}$$

For a proper definition of $\mathcal{H}$, see (4) below.

It turns out that getting an a priori estimate in H^3 is the crucial part of the work, this is the content of Theorem 4. We achieve that by a bootstrapping argument, where we use the constant variation formula representation of the solution. On the other hand the H^2 estimates are much easier to establish. We take advantage of the boundedness of the domain and availability of the Sobolev inequalities. It turns out that we cannot repeat this part of the argument on an unbounded domain, e.g. $\mathbb{R}^2$.

Once we set the objectives, we describe the methods to achieve that goal. We use the notation and the guidance of the semigroup theory, see [3]. From our perspective, problem (1) does not justify the full-fledged theory. We choose an easier approach that bases on Fourier series.

Here, we are content with establishing global in time existence. We do not study here the asymptotic behavior of the system. We postpone it for a future work.

We should also mention, that [6], [7] and [8] are the only closely related papers we are aware of. In [6] the authors are concerned with the one-dimensional version of the same problem. However, the approach applied there is completely different, for the authors use the Galerkin method. This general tool is not best suited for the regularity study, so that they have to overcome additional technical difficulties which are absent here, in their uniqueness result. Moreover, [6] presents also numerical results on coarsening and stationary states.

The other papers are [7] and [8]. The authors study a similar sixth order problem, which also belongs to a class of Cahn-Hilliard equations. The motivation to study that problem comes from a different physical phenomenon, namely the phase transitions in ternary oil-water-surfactant systems considered in a bounded domain.

They obtain similar results by different methods, i.e. the typical tools of the theory of parabolic equations due to Solonnikov [9].

Notation We will clarify the notation we use. We identify the flat torus $\mathbb{T}^2$ with $[0,2\pi)^2$, (x,y) is a generic point of $\mathbb{T}^2$. By $\mathrm{d}V = \mathrm{d}x\mathrm{d}y$ we denote the Lebesgue measure. For $h:\mathbb{T}^2\to\mathbb{R}$, we will write

$$\|h\| = \|h\|_{L_2(\mathbb{T}^2)}, \quad \|\nabla h\| = \left(\int_{\mathbb{T}^2} ((h_x)^2 + (h_y)^2)\,\mathrm{d}V\right)^{1/2}.$$

Since we work on the torus, in place of the Fourier transform we consider the Fourier series, which may be written formally as

$$h(x,y) = \sum_{(k,l)\in\mathbb{Z}^2} e^{-i(xk+yl)}\hat{h}(k,l) = \int_{\mathbb{R}^2} e^{-i(xk+yl)}\hat{h}(k,l)\,\mathrm{d}\mu(k,l),$$

where μ is the standard counting measure supported on $\mathbb{Z}^2$. In this formula we use

$$\hat{h}(k,l) = \frac{1}{(2\pi)^2}\int_{\mathbb{T}^2} h(x,y)e^{i(xk+yl)}\,\mathrm{d}V(x,y).$$

For the sake of consistency we also recall the inverse Fourier transform for $f:\mathbb{Z}^2\to\mathbb{R}$. Namely, we define

$$\check{f}(x,y) = \sum_{(k,l)\in\mathbb{Z}^2} e^{-i(xk+yl)}f(k,l).$$

Moreover, we notice that for any $s\in\mathbb{R}$, the norm in the Sobolev space $H^s(\mathbb{T}^2)$ is equivalent to

$$\|f\|_{H^s(\mathbb{T}^2)} = \|(1+|\cdot|^2)^{s/2}\hat{f}(\cdot)\|_{L_2(\mu)}.$$

2 Local in time existence

We want to discover as much structure of (1) as possible. For this purpose we define a vector field

$$F = \frac{\alpha}{3}(h_x^3, h_y^3) + \beta(h_y^2 h_x, h_x^2 h_y)$$

and the functions

$$\Psi = \beta(h_y^2 h_{xx} + h_x^2 h_{yy} + 4h_x h_y h_{xy}) + \alpha(h_x^2 h_{xx} + h_y^2 h_{yy}),$$

$$\Phi = \frac{1}{2}(h_{xx}+h_{yy})^2 - \frac{1}{2}(h_x^2+h_y^2) + \frac{\alpha}{12}\left(h_x^4+h_y^4\right) + \frac{\beta}{2}h_x^2 h_y^2.$$

Note that $\operatorname{div} F = \Psi$. Subsequently we shall write

$$\mathcal{H} = \Delta h + \Delta^2 h - \Psi = \operatorname{div}\,(\nabla h + \nabla\Delta h - F). \tag{4}$$

Thus, indeed (1) takes the form (3). We notice that due to the periodic boundary condition the average of $\mathcal{H}$ vanishes, $\int_{\mathbb{T}^2}\mathcal{H}\,\mathrm{d}V = 0$.

Finally, we define the functional

$$\mathcal{L} = \int_{\mathbb{T}^2} \Phi \, \mathrm{d}V. \tag{5}$$

The first stage of our analysis of (1) is a study of the following linear equation

$$h_t = \Delta^3 h + f, \qquad h(0,\cdot) = h_0(\cdot), \tag{6}$$

where $f : \mathbb{T}^2 \to \mathbb{R}$ is a given function whose regularity has to be specified yet. Although we first treat (6), we keep in mind that we finally want to consider

$$f(h) = \frac{D}{2}|\nabla h|^2 + \Delta^2 h - \Delta\Psi(h). \tag{7}$$

We proceed formally by applying the Fourier transform to both sides, this yields,

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{h}(t,\xi) = -|\xi|^6 \hat{h}(t,\xi) + \hat{f}, \quad \hat{h}(0,\xi) = \hat{h_0}(\xi).$$

After solving this ODE we obtain an explicit formula for the Fourier transform of the solution,

$$\hat{h}(t,\xi) = e^{-|\xi|^6 t}\hat{h_0}(\xi) + \int_0^t e^{-|\xi|^6(t-s)}\hat{f}(s,\xi)\,\mathrm{d}s.$$

Thus, we can write

$$h(t,(x,y)) = \left(e^{-|\xi|^6 t}\hat{h_0}(\xi) + \int_0^t e^{-|\xi|^6(t-s)}\hat{f}(s,\xi)\,\mathrm{d}s\right)^{\vee}(x,y).$$

After introducing the following shorthand

$$e^{\Delta^3 t} f = \left(e^{-|\cdot|^6 t}\hat{f}(\cdot),\right)^{\vee} \tag{8}$$

we can write a solution of (6) in the form:

$$h(t) = e^{\Delta^3 t}h_0 + \int_0^t e^{\Delta^3(t-s)} f(s)\,\mathrm{d}s.$$

Once we derived the above constant variation formula for solutions to (6), we introduce the operator

$$\mathcal{F}(h)(t,\cdot) = (e^{\Delta^3 t}h_0)(\cdot) + \int_0^t e^{\Delta^3(t-s)} f(h(s,\cdot))\,\mathrm{d}s \tag{9}$$

with f given by (7). We notice that the above $\mathcal{F}$ is well-defined on the following space

$$X_T = C\left([0,T]; H^3(\mathbb{T}^2)\right).$$

The ball centered at zero with radius M will be denoted by X_T^M,

$$X_T^M = X_T \cap \{v : \sup_{t\in[0,T]} ||v(t)||_{H^3} \le M\}.$$

Theorem 2 *Let us assume that $h_0 \in H^3$ and let us fix $M/2 > ||h_0||_{H^3}$. Then, there exists $T > 0$ such that $\mathcal{F} : X_T^M \to X_T^M$ and $\mathcal{F}$ is a contraction on X_T^M. In particular, there exists a unique solution of the integral equation $\mathcal{F}(h) = h$ in X_T^M.*

Remark. The solution constructed in the above theorem will be called a *mild solution* to (1).

Proof. We shall write L_2 for $L_2(\mu)$, where μ is the counting measure. For any $s \in \mathbb{R}$ we will use $H^s = H^s(\mathbb{T}^2)$. We shall first check that the operator defined by (8) is continuous on H^s for any s and all $t > 0$. Indeed,

$$\begin{aligned} \|e^{\Delta^3 t} h_0\|_{H^s} &= \|(1+|\xi|^2)^{s/2}(e^{\Delta^3 t}h_0)^\wedge(\xi)\|_{L_2} = \|(1+|\xi|^2)^{s/2} e^{-|\xi|^6 t}\hat{h_0}(\xi)\|_{L_2} \\ &\leq \|(1+|\xi|^2)^{s/2}\hat{h_0}(\xi)\|_{L_2} = \|h_0\|_{H^s}. \end{aligned}$$

We also want to use continuity of the function, $t \mapsto e^{\Delta^3 t}h_0 \in C([0,T]; H^s)$. It follows from the Lebesgue's dominated convergence theorem, namely

$$\lim_{t\to t_0} \|(e^{\Delta^3 t} - e^{\Delta^3 t_0})h_0\|_{H^s} = \lim_{t\to t_0} \|(1+|\xi|^2)^{s/2}(e^{-|\xi|^6 t} - e^{-|\xi|^6 t_0})\hat{h_0}(\xi)\|_{L_2} = 0.$$

We shall establish a regularizing property of $\mathcal{F}$ which is a crucial point in our theory. We claim that for any $p \in \mathbb{R}$, $0 < \varepsilon$, $0 \leq t_0 \leq t$ and a function $v \in C([t_0,t], H^{p-6(1-\varepsilon)})$ we have

$$\| \int_{t_0}^t e^{\Delta^3(t-s)} v(s,\cdot)\, \mathrm{d}s \|_{H^p} \leq C(\varepsilon)\; e^t (t-t_0)^\varepsilon \; \|v\|_{C([0,t];H^{p-6(1-\varepsilon)})}. \tag{10}$$

Indeed, let us notice

$$\begin{aligned} \| \int_{t_0}^t e^{\Delta^3(t-s)} v(s,\cdot)\, \mathrm{d}s \|_{H^p} &= \|(1+|\xi|^2)^{p/2} \left(\int_{t_0}^t e^{\Delta^3(t-s)} v(s,\cdot)\, \mathrm{d}s \right)^\wedge (\xi)\|_{L_2} \\ &= \|(1+|\xi|^2)^{p/2} \int_{t_0}^t e^{-|\xi|^6(t-s)}\hat{v}(s,\xi)\, \mathrm{d}s\|_{L_2} \\ &\leq \int_{t_0}^t \|(1+|\xi|^2)^{p/2} e^{-|\xi|^6(t-s)}\hat{v}(s,\xi)\|_{L_2}\, \mathrm{d}s. \end{aligned}$$

At this point we make a simple observation, for $t > s > 0$

$$-|\xi|^6(t-s) \leq t - (1+|\xi|^6)(t-s) \leq t - \frac{1}{4}(1+|\xi|^2)^3(t-s).$$

As a result, for any $\varepsilon \in (0,1]$ we have

$$\| \int_{t_0}^t e^{\Delta^3(t-s)} v(s,\cdot)\, \mathrm{d}s \|_{H^p} \leq$$

$$e^t \int_{t_0}^t \|e^{-\frac{1}{4}(1+|\xi|^2)^3(t-s)}(t-s)^{1-\varepsilon}(1+|\xi|^2)^{3(1-\varepsilon)} \frac{1}{(t-s)^{1-\varepsilon}} (1+|\xi|^2)^{\frac{p}{2}-3(1-\varepsilon)}\hat{v}(s,\xi)\|_{L_2}\, \mathrm{d}s.$$

If $y = (1+|\xi|^2)^3(t-s)$, then

$$e^{-\frac{1}{4}(1+|\xi|^2)^3(t-s)}(t-s)^{1-\varepsilon}(1+|\xi|^2)^{3(1-\varepsilon)} = e^{-\frac{1}{4}y} y^{1-\varepsilon} \leq C(\varepsilon),$$

where $C(\varepsilon)$ is a constant that may vary during the proof. Therefore,

$$\begin{aligned}\| \int_{t_0}^{t} e^{\Delta^3(t-s)} v(s,\cdot)\,\mathrm{d}s \|_{H^p} &\leq e^t C(\varepsilon) \int_{t_0}^{t} \| \frac{1}{(t-s)^{1-\varepsilon}} (1+|\xi|^2)^{\frac{p}{2}-3(1-\varepsilon)} \hat{v}(s,\xi) \|_{L_2}\,\mathrm{d}s \\ &\leq e^t \frac{C(\varepsilon)}{\varepsilon} (t-t_0)^{\varepsilon} \sup_{s\in[0,t]} \|v(s,\cdot)\|_{H^{p-6(1-\varepsilon)}}.\end{aligned}$$

Thus, we have derived (10).

Subsequently, we take $p = 3$ and we consider (10) with $t_0 = 0$. In order to prove that $\mathcal{F}$ maps X_T into X_T one has to verify that for any $h \in X_T^M$, the following bound holds

$$\sup_{t\in[0,T]} \|f(h(t,\cdot))\|_{H^{3-6(1-\varepsilon)}} \leq C(M) < \infty, \tag{11}$$

where $C(M)$ is independent of h.

We select $0 < \varepsilon < 1/3$. Obviously, by the definition of the norm and our choice of ε, we see that

$$\|\Delta^2 h\|_{H^{3-6(1-\varepsilon)}} \leq C\|h\|_{H^{7-6(1-\varepsilon)}} \leq C\|h\|_{H^3}.$$

Since the embedding

$$H^2(\mathbb{T}^2) \hookrightarrow C(\mathbb{T}^2) \cap L_\infty(\mathbb{T}^2) \tag{12}$$

is valid (see [1]), then for any element $h \in X_T^M$ we have

$$\begin{aligned}\|h_x^2\|^2_{H^{3-6(1-\varepsilon)}} \leq \|h_x^2\|^2_{L_2} = \int_{\mathbb{T}^2} h_x^4\,\mathrm{d}V \leq \|h_x\|^2_\infty \int_{\mathbb{T}^2} h_x^2\,\mathrm{d}V \leq C\|h\|^2_{H^3}\|h_x\|^2_{L_2} \\ \leq C\|h\|^4_{H^3} \leq CM^4.\end{aligned}$$

We conclude that

$$\sup_{t\in[0,T]} \||\nabla h(t)|^2\|_{C([0,t];H^{3-6(1-\varepsilon)})} \leq CM^2.$$

Finally, if we restrict ε even further by requiring that $\varepsilon < 1/6$, then we have the following estimate for the nonlinearity,

$$\begin{aligned}\|\Delta(h_x h_y h_{xy})\|_{H^{3-6(1-\varepsilon)}} &\leq C\|h_x h_y h_{xy}\|_{H^{5-6(1-\varepsilon)}} \leq C\|h_x h_y h_{xy}\|_{L_2} \\ &\leq C\|h_x\|_\infty \|h_y\|_\infty \|h_{xy}\|_{L_2} \leq C\|h\|^3_{H^3}.\end{aligned}$$

After combining these observation, we conclude that

$$\sup_{t\in[0,T]} \|\Delta\Psi(h)\|_{C([0,t];H^{3-6(1-\varepsilon)})} \leq C(M^2 + M^3).$$

This implies that $\mathcal{F} : X_T^M \to X_T^M$, where T is so chosen, that for given M we have $C(\varepsilon)e^T T^\varepsilon (M + M^2 + M^3) < M/2$.

Our next goal is to prove that $\mathcal{F} : X_T^M \to X_T^M$ is a contraction for sufficiently small $T > 0$. For this purpose, because of (10) it is enough to show that f is Lipschitz continuous in X_T^M,

$$\|f(v) - f(u)\|_{C([0,t];H^{3-6(1-\varepsilon)})} \leq C(M)\|u - v\|_{C([0,t];H^3)} \tag{13}$$

for a positive $\varepsilon \in (0, 1/3)$. Once we establish (13), taking $e^T T^\varepsilon < \frac{1}{2C(M)C(\varepsilon)}$ will finish the proof.

Now we show (13). Here the linear term $\Delta^2 v$ does not cause any problems, while some more work has to be invested for the nonlinearities. In order to deal with the term $|\nabla v|^2$, we observe that for $\varepsilon < 1/2$ the number $s = 3 - 6(1-\varepsilon)$ is negative. Therefore,

$$\|u_x^2 - v_x^2\|_{H^s} \le \|u_x^2 - v_x^2\|_{L_2} \le C\|u_x - v_x\|_{L_\infty}\|u_x + v_x\|_{L_\infty} \le CM\|u - v\|_{H^3}.$$

In the above estimates we used the embedding (12). In order to finish the proof we consider the nonlinear term $\Delta(v_x v_y v_{xy})$. We have

$$\begin{aligned}
&\|\Delta u_x u_y u_{xy} - \Delta v_x v_y v_{xy}\|_{H^s} \le \|u_x u_y u_{xy} - v_x v_y v_{xy}\|_{H^{s+2}} \\
&\quad \le \quad \|(u_x - v_x)u_y u_{xy}\|_{H^{s+2}} + \|v_x u_{xy}(u_y - v_y)\|_{H^{s+2}} + \|v_x v_y(u_{xy} - v_{xy})\|_{H^{s+2}}.
\end{aligned}$$

Note that for $\varepsilon \in (0, 1/6)$ we have $s + 2 < 0$, hence $\|\cdot\|_{H^{s+2}} \le C\|\cdot\|_{L_2}$. Therefore

$$\begin{aligned}
\|(u_x - v_x)u_y u_{xy}\|_{H^{s+2}} &\le C\|(u_x - v_x)u_y u_{xy}\|_{L_2} \le C\|u_x - v_x\|_\infty \|u_y\|_\infty \|u_{xy}\|_{L_2} \\
&\le CM^2\|u - v\|_{H^3}
\end{aligned}$$

and similarly

$$\|v_x u_{xy}(u_y - v_y)\|_{H^{s+2}} \le CM^2\|u - v\|_{H^3}.$$

Finally, we have

$$\begin{aligned}
\|v_x v_y(u_{xy} - v_{xy})\|_{H^{s+2}} &\le C\|v_x v_y(u_{xy} - v_{xy})\|_{L_2} \le C\|v_x\|_\infty \|v_y\|_\infty \|u_{xy} - v_{xy}\|_{L_2} \\
&\le CM^2\|u - v\|_{H^3}.
\end{aligned}$$

The same technique may be used to estimate the other two terms. We have derived (13). □

Once we have established existence of a unique fixed point of $\mathcal{F}$, we will prove that the solution of the equation $\mathcal{F}(h) = h$ enjoys some additional regularity. Namely, any fixed point is locally Hölder continuous in the norm $\|\cdot\|_{H^3(\mathbb{T}^2)}$ with respect to time.

Lemma 1 *Let us take any $p \in \mathbb{R}$. For every $0 < a \le 1$ there exists a constant $C_a > 0$ such that for $\delta > 0$*

$$\|(e^{\Delta^3 \delta} - Id)g\|_{H^p} \le \frac{C_a}{a}\delta^a \|g\|_{H^{6a+p}}.$$

Proof. We begin with an observation about the exponential function. Namely, there exists a constant C_a such that for $x \ge 0$ we have

$$1 - e^{-x} \le \frac{C_a}{a}x^a.$$

Indeed, for $x = 0$ both sides are equal, hence it is enough to show the inequality for the derivatives $e^{-x} \le C_a x^{a-1}$ for some $C_a > 0$. But this is obvious, since for $a = 1$ we have $e^{-x} \le 1$ and for $a \in (0, 1)$ the function $(0, \infty) \ni x \mapsto e^x x^{a-1}$ has infinite limits when $x \to 0^+$ and $x \to \infty$.

We use this observation in the following estimate,

$$\|(e^{\Delta^3\delta}-I)g\|_{H^p} = \|(e^{-|\xi|^6\delta}-1)\hat{g}(1+|\xi|^2)^{p/2}\|_{L_2} \le \frac{C_a}{a}\delta^a\|\hat{g}|\xi|^{6a}(1+|\xi|^2)^{p/2}\|_{L_2}$$
$$\le \frac{C_a}{a}\delta^a\|g\|_{H^{6a+p}}. \qquad \square$$

Now we can show better regularity of the fixed point constructed in the previous theorem. Here is the first step in this direction.

Lemma 2 *The unique solution of the equation $\mathcal{F}(h) = h$, where $\mathcal{F}$ is given by formula (9), is locally Hölder continuous in the norm $\|\cdot\|_{H^3(\mathbb{T}^2)}$ with respect to time. More precisely, there exist constants $a, \varepsilon_1 > 0$ such that*

$$\|h(t+\delta)-h(t)\|_{H^3} \le C(\delta t^{-1} + \delta^a t^{\varepsilon_1} + \delta^{\varepsilon_1})$$

for a constant $C = C(\varepsilon_1, M, a)$.

Proof. We have the following estimate

$$\begin{aligned}\|h(t+\delta,\cdot)-h(t,\cdot)\|_{H^3} \le\ & \|(e^{\Delta^3\delta}-I)e^{\Delta^3 t}h_0\|_{H^3}\\ &+\|\int_0^t (e^{\Delta^3\delta}-I)e^{\Delta^3(t-s)}f(h(s,\cdot))\,\mathrm{d}s\|_{H^3}\\ &+\|\int_t^{t+\delta} e^{\Delta^3(t+\delta-s)}f(h(s,\cdot))\,\mathrm{d}s\|_{H^3}.\end{aligned}$$

We observe that the first term on the RHS can be bounded as follows,

$$\begin{aligned}\|(e^{\Delta^3 t+\delta}-e^{\Delta^3 t})h_0\|_{H^3} &= \|(1+|\xi|^2)^{3/2}e^{-|\xi|^6 t}(1-e^{-|\xi|^6\delta})\widehat{h_0}\|_{L_2}\\ &\le C\|(1+|\xi|^2)^{3/2}e^{-|\xi|^6 t}|\xi|^6 t\delta\frac{1}{t}\widehat{h_0}\|_{L_2}\\ &\le C\frac{\delta}{t}\|(1+|\xi|^2)^{3/2}\widehat{h_0}\|_{L_2} = C\frac{\delta}{t}\|h_0\|_{H^3} \le CM\frac{\delta}{t}.\end{aligned}$$

This means that the first term is even locally Lipschitz continuous. From (10) and (11) we deduce

$$\|\int_t^{t+\delta} e^{\Delta^3(t+\delta-s)}f(h(s,\cdot))\,\mathrm{d}s\|_{H^3} \le C(\varepsilon)M\delta^\varepsilon.$$

Finally, using Lemma 1 for any positive a and formula (10) with $t_0 = 0$ and any $\varepsilon_1 > 0$, we obtain

$$\begin{aligned}\|\int_0^t (e^{\Delta^3\delta}-I)e^{\Delta^3(t-s)}f(h(s,\cdot))\,\mathrm{d}s\|_{H^3} &\le \int_0^t C_a\frac{\delta^a}{a}\|e^{\Delta^3(t-s)}f(h(s,\cdot))\|_{H^{3+6a}}\,\mathrm{d}s\\ &\le C_a\frac{\delta^a}{a}t^{\varepsilon_1}\|f(h)\|_{C([0,t];H^{3+6a-6(1-\varepsilon_1)})}.\end{aligned}$$

Once we apply (11) with $a+\varepsilon_1 < \varepsilon < 1/6$ to the above term, we will come to the desired conclusion, i.e.

$$\|\int_0^t (e^{\Delta^3\delta}-I)e^{\Delta^3(t-s)}f(h(s,\cdot))\,\mathrm{d}s\|_{H^3} \le C_a\frac{\delta^a}{a}t^{\varepsilon_1}C(M). \qquad \square$$

Next is our regularity theorem, which explains that h, the mild solution to (1), is in fact a *weak solution* to (1), in the sense that $h \in C([0,T];H^3)$ and $h_t \in C((0,T);H^{-3})$ and the equation is satisfied in the distributional sense.

Theorem 3 *The solution $h \in X_T^M$ of the integral equation $\mathcal{F}(h) = h$ is differentiable with respect to time in the H^{-3} norm and*

$$h_t(t,\cdot) = \Delta^3 h(t,\cdot) + f(h(t,\cdot))$$

in the distributional sense, with initial condition $h(0,\cdot) = h_0(\cdot)$. As a result, it is a weak solution of (1).

Proof. We shall show that h is a limit (in the $C^1([a, T-a]; H^3)$ norm) of functions with the desired property. This approach was used in the proof of [3, Lemma 3.2.1].

For $t > \delta > 0$ we define

$$h^\delta(t,\cdot) = e^{\Delta^3 t} h_0(\cdot) + \int_0^{t-\delta} e^{\Delta^3(t-s)} f(h(s,\cdot))\,\mathrm{d}s.$$

Then,

$$\frac{\mathrm{d}h^\delta}{\mathrm{d}t}(t,\cdot) = \Delta^3 e^{\Delta^3 t} h_0(\cdot) + e^{\Delta^3 \delta} f(h(t-\delta,\cdot)) + \int_0^{t-\delta} \Delta^3 e^{\Delta^3(t-s)} f(h(s,\cdot))\,\mathrm{d}s,$$

where we treat the above functions like elements of $H^{-3}(\mathbb{T}^2)$. Indeed, using our standard arguments we notice

$$\|\Delta^3 e^{\Delta^3 t} h_0(\cdot)\|_{H^{-3}} < CM, \qquad \|e^{\Delta^3 \delta} f(h(t-\delta,\cdot))\|_{H^{-3}} < CM.$$

Moreover, for any $s \in \mathbb{R}$

$$\|\Delta^3 e^{\Delta^3 t} g(\cdot)\|_{H^s} \le \||\xi|^6 e^{-|\xi|^6 t}(1+|\xi|^2)^{s/2}\widehat{g}(\cdot)\|_{L_2} \le \frac{C}{t}\|(1+|\xi|^2)^{\frac{s}{2}}\widehat{g}(\cdot)\|_{L_2} = \frac{C}{t}\|g\|_{H^s}.$$

Hence the norm of $\Delta^3 e^{\Delta^3 t}$ in $L(H^s, H^s)$ may be bounded by C/t. As a result we arrive at

$$\begin{aligned}
\|\int_0^{t-\delta} \Delta^3 e^{\Delta^3(t-s)} f(h(s,\cdot))\,\mathrm{d}s\|_{H^{-3}} &\le \sup_{s\in[0,t-\delta]} \|f(h(s,\cdot))\|_{H^{-3}} \int_0^{t-\delta} \frac{1}{t-s}\,\mathrm{d}s \\
&\le \sup_{s\in[0,t-\delta]} \|f(h(s,\cdot))\|_{H^{3-6(1-\varepsilon)}} \ln|\delta/t| \\
&\le C(M)\ln|\delta/t| < \infty.
\end{aligned}$$

We have

$$\frac{\mathrm{d}h^\delta}{\mathrm{d}t}(t,\cdot) = e^{\Delta^3\delta} f(h(t-\delta,\cdot)) + \Delta^3 h^\delta(s,\cdot).$$

In order to finish the proof we have to show that

$$\|h^\delta(t,\cdot) - h(t,\cdot)\|_{H^{-3}} \xrightarrow[\delta\to 0]{} 0,$$

$$e^{\Delta^3\delta} f(h(t-\delta,\cdot)) \xrightarrow[\delta\to 0]{\|\cdot\|_{H^{-3}}} f(h(t,\cdot)), \quad \Delta^3 h^\delta(s,\cdot) \xrightarrow[\delta\to 0]{\|\cdot\|_{H^{-3}}} \Delta^3 h(s,\cdot)$$

and use the limit differentiation theorem.

Our first observation is

$$\begin{aligned}
\|h(t,\cdot) - h^\delta(t,\cdot)\|_{H^3} &= \|\int_{t-\delta}^{t} e^{\Delta^3(t-s)} f(h(s,\cdot))\,\mathrm{d}s\|_{H^3} \\
&\le C(T,\varepsilon)\delta^\varepsilon \sup_{s\in[0,t]} \|f(h(s,\cdot))\|_{H^{3-6(1-\varepsilon)}} \le C(T,M)\delta^\varepsilon \xrightarrow[\delta\to 0]{} 0.
\end{aligned}$$

Secondly, we note

$$\begin{aligned}\|e^{\Delta^3\delta}f(h(t-\delta,\cdot))-f(h(t,\cdot))\|_{H^{-3}} &\le \|(e^{\Delta^3\delta}-Id)f(h(t-\delta,\cdot))\|_{H^{-3}}\\ &+ \|f(h(t-\delta,\cdot))-f(h(t,\cdot))\|_{H^{-3}}.\end{aligned}$$

Due to (13), we arrive at

$$\|f(h(t-\delta,\cdot))-f(h(t,\cdot))\|_{H^{-3}} \le C(M)\|h(t-\delta,\cdot)-h(t,\cdot)\|_{H_3} \xrightarrow[\delta\to 0]{} 0,$$

because $h\in C([0,T];H^3)$. Moreover, using Lemma 1 we have

$$\|(e^{\Delta^3\delta}-\mathbb{I})f(h(t-\delta,\cdot))\|_{H^{-3}} \le \frac{C_b}{b}\delta^b\|f(h(t-\delta,\cdot))\|_{H^{6b-3}} \le C\frac{C_b}{b}\delta^b \xrightarrow[\delta\to 0]{} 0,$$

because $6b-3\le 3-6(1-\varepsilon)$ for sufficiently small $b>0$. Finally, Theorem 2 implies that,

$$\begin{aligned}\|\Delta^3 h_\delta(t,\cdot) &- \Delta^3 h(t,\cdot)\|_{H^{-3}} = \|\int_{t-\delta}^t \Delta^3 e^{\Delta^3(t-s)}f(h(s,\cdot))\,\mathrm{d}s\|_{H^{-3}}\\ &= \|\int_{t-\delta}^t \Delta^3 e^{\Delta^3(t-s)}(f(h(s,\cdot))-f(h(t,\cdot)))\,\mathrm{d}s + \int_{t-\delta}^t \Delta^3 e^{\Delta^3(t-s)}f(h(t,\cdot)))\,\mathrm{d}s\|_{H^{-3}}\\ &\le \int_{t-\delta}^t \|\Delta^3 e^{\Delta^3(t-s)}\|_{(H^{-3}\to H^{-3})}\|f(h(s,\cdot))-f(h(t,\cdot))\|_{H^{-3}}\,\mathrm{d}s\\ &\quad +\|\int_{t-\delta}^t -\frac{\mathrm{d}}{\mathrm{d}s}\left(e^{\Delta^3(t-s)}f(h(t,\cdot)))\right)\mathrm{d}s\|_{H^{-3}}\\ &\le \int_{t-\delta}^t \frac{C}{t-s}\|h(t,\cdot)-h(s,\cdot)\|_{H^3}\,\mathrm{d}s + \|(e^{\Delta^3\delta}-\mathbb{I})f(h(t,\cdot))\|_{H^{-3}}\\ &\le \int_{t-\delta}^t \frac{C(T)}{t-s}\left((t-s)^\theta + \frac{t-s}{t-\delta}\right)\mathrm{d}s + C\frac{C_a}{a}\delta^a\\ &= \frac{C}{\theta}C(T)\left(\frac{\delta^{\varepsilon_1}}{\varepsilon_1} + \frac{\delta}{t-\delta}\frac{C_a}{a}\delta^a\right) \xrightarrow[\delta\to 0]{} 0.\end{aligned}$$

Moreover, the convergence is uniform for t in compact subsets of $(0,T)$. □

3 *A priori* estimates, global existence

In this Section we derive an *a priori* estimate in the space $L_2([0,T];H^3(\mathbb{T}^2))$. Before we present this main result, let us prove a useful bound

Lemma 3 *For $\rho,\tau\ge 0$ we have*

$$\sup_{y\ge 0}(1+y)^\tau e^{-\rho y^3} \le C(\tau)\max\{1,\rho^{-\tau/3}\}. \tag{14}$$

Proof. If $y\le 1$ then

$$(1+y)^\tau e^{-\rho y^3} \le 2^\tau$$

and if $y\ge 1$ then

$$(1+y)^\tau e^{-\rho y^3} \le (2y)^\tau e^{-\rho y^3} = 2^\tau\rho^{-\tau/3}(\rho y^3)^{\tau/3}e^{-\rho y^3} \le C(\tau)\rho^{-\tau/3}. \; □$$

Theorem 4 *Let us assume that h is a weak solution to (1) and (2), which was constructed in Theorem 2. In addition, we assume that $h_0 \in H^3$. Then, $h \in L_\infty(0,T;H^3)$ and*

$$\|h\|_{L_\infty(0,T;H^3)} \leq C_3(h_0,T),$$

where the constant $C(h_0,T)$ depends only of T and the initial data h_0.

Proof. *Step 1.* Differentiating $\mathcal{L}$ with respect to time (see (5)) and integrating by parts we obtain

$$\begin{aligned}\frac{\mathrm{d}\mathcal{L}}{\mathrm{d}t} &= \int_{\mathbb{T}^2} \left(\Delta h \Delta h_t - (h_x h_{xt} + h_y h_{yt}) + \frac{\alpha}{3}\left(h_x^3 h_{xt} + h_y^3 h_{yt}\right) + \beta\left(h_y^2 h_x h_{xt} + h_x^2 h_y h_{yt}\right)\right) \mathrm{d}V \\ &= \int_{\mathbb{T}^2} \left(\Delta^2 h + (h_{xx} + h_{yy}) - \alpha\left(h_x^2 h_{xx} + h_y^2 h_{yy}\right) - \beta\left(4h_x h_y h_{xy} + h_y^2 h_{xx} + h_x^2 h_{yy}\right)\right) h_t \,\mathrm{d}V \\ &= \int_{\mathbb{T}^2} \mathcal{H} h_t \,\mathrm{d}V.\end{aligned}$$

Thus, since h is a weak solution of (1), then

$$\frac{\mathrm{d}\mathcal{L}}{\mathrm{d}t} = \int_{\mathbb{T}^2} \mathcal{H}\left(\frac{D}{2}|\nabla h|^2 + \Delta\mathcal{H}\right) \mathrm{d}V = -\int_{\mathbb{T}^2} |\nabla\mathcal{H}|^2 \,\mathrm{d}V + \frac{D}{2}\int_{\mathbb{T}^2} \mathcal{H}|\nabla h|^2 \,\mathrm{d}V.$$

Since $\int_{\mathbb{T}^2} \mathcal{H}\,\mathrm{d}V = 0$, we have the Sobolev inequality

$$\int_{\mathbb{T}^2} \mathcal{H}^2 \,\mathrm{d}V \leq 2\pi \int_{\mathbb{T}^2} |\nabla\mathcal{H}|^2 \,\mathrm{d}V.$$

Moreover,

$$\frac{D}{2}\mathcal{H}|\nabla h|^2 \leq \frac{1}{2\pi}\mathcal{H}^2 + \frac{\pi D^2}{8}|\nabla h|^4.$$

As a result,

$$\frac{\mathrm{d}\mathcal{L}}{\mathrm{d}t} \leq \frac{\pi D^2}{8}\int_{\mathbb{T}^2} |\nabla h|^4 \,\mathrm{d}V \leq C_1 + C_2\mathcal{L}, \tag{15}$$

where $C_i = C_i(D,\alpha,\beta) > 0, i = 1,2$, because we can find $D_i = D_i(\alpha,\beta) > 0, i = 1,2$ such that

$$|\nabla h|^4 \leq D_1 + D_2\left(-\frac{1}{2}(h_x^2 + h_y^2) + \frac{\alpha}{12}\left(h_x^4 + h_y^4\right) + \frac{\beta}{2}h_x^2 h_y^2\right) \leq D_1 + D_2\Phi.$$

Due to the Gronwall inequality we deduce from (15) that

$$\mathcal{L}(t) \leq \left(\frac{C_1}{C_2} + \mathcal{L}(0)\right)e^{C_2 t} - \frac{C_1}{C_2},$$

so h is bounded in $L_\infty([0,T];H^2(\mathbb{T}^2))$ for a fixed $T < \infty$. Let us notice that this bound is not uniform with respect to $T > 0$.

We keep the following observation in mind,

$$K^{-1}\|u\|_{H^{2\alpha}} \leq \|(Id - \Delta)^\alpha u\|_{L_2} \leq K\|u\|_{H^{2\alpha}}. \tag{16}$$

It will be used below.

Step 2. If $\alpha < \frac{3}{2}$, then

$$\|h\|_{L_\infty(0,T;H^{2\alpha})} \le C_{2\alpha}(h_0, T).$$

In order to show this bound we apply $(Id - \Delta u)^\alpha$ to both sides of the constant variation formula

$$h(t) = e^{\Delta^3 t} h_0(\cdot) + \int_0^t e^{\Delta^3(t-s)} f(h(s,\cdot))s, \tag{17}$$

where f is given by (7). Taking the L_2 norms yields,

$$\begin{aligned}
\|h\|_{H^{2\alpha}} \le \|h_0\|_{H^{2\alpha}} &+ \frac{DK}{2}\int_0^t \|(Id-\Delta)^\alpha e^{\Delta^3(t-s)}|\nabla h(s,\cdot)|^2\| \,\mathrm{d}s \\
&+ K\int_0^t \|(Id-\Delta)^{\alpha+1} e^{\Delta^3(t-s)}\Delta h\| \,\mathrm{d}s + K\int_0^t \|(Id-\Delta)^{\alpha} e^{\Delta^3(t-s)}\Delta h\| \,\mathrm{d}s \\
&+ K\int_0^t \|(Id-\Delta)^{\alpha+1} e^{\Delta^3(t-s)}\mathrm{div}\, F\| \,\mathrm{d}s + K\int_0^t \|(Id-\Delta)^{\alpha} e^{\Delta^3(t-s)}\mathrm{div}\, F\| \,\mathrm{d}s \\
&= \|h_0\|_{H^{2\alpha}} + I_1 + I_2 + I_3 + I_4 + I_5,
\end{aligned}$$

where $I_k, k = 1, \dots, 5$ are ordered abbreviations for the five time integral terms. We have $I_3 \le I_2$ and $I_5 \le I_4$. We will estimate separately the terms I_1, I_2 and I_4.

With (14) it is easy to estimate I_2,

$$\begin{aligned}
|I_2| &= K\int_0^t \|(1+|\cdot|^2)^{1+\alpha} e^{-|\cdot|^6(t-s)}(\Delta h)^\wedge(s,\cdot)\| \,\mathrm{d}s \\
&\le C(\alpha)\, \mathrm{essup}_{t\in[0,T]}\|h\|_{H^2}(t)\int_0^t \max\{1, (t-s)^{-(1+\alpha)/3}\} \,\mathrm{d}s \le C_2(h_0,T) < \infty.
\end{aligned}$$

Here we use $(1+\alpha)/3 < 1$.

We shall deal with a representative term h_x^3 in I_4, estimates for other three terms h_y^3, $h_y^2 h_x$, $h_x^2 h_y$ in F are similar,

$$\begin{aligned}
\frac{K}{3}\int_0^t (t-s)^{-\frac{\alpha+3/2}{3}}\|h_x^3\| \,\mathrm{d}s &\le \frac{K}{3}\int_0^t (t-s)^{-\frac{\alpha+3/2}{3}}\|h_x\|_{L_6}^3 \,\mathrm{d}s \\
&\le CK\left(\mathrm{essup}_{t\in[0,T]}\|h\|_{H^2}(t)\right)^3 \int_0^t (t-s)^{-\frac{\alpha+3/2}{3}} \,\mathrm{d}s \le C_4(h_0,T) < \infty.
\end{aligned}$$

We used here the assumption that $\alpha < 3/2$ and the two-dimensional Sobolev embedding

$$\|\nabla h\|_{L_p(\mathbb{T}^2)} \le C\|\nabla^2 h\|_{L_2(\mathbb{T}^2)}, \qquad p < \infty.$$

We estimate I_1 as follows,

$$\begin{aligned}
I_1 \le C(\mathrm{essup}_{t\in[0,T]}\||\nabla h|^2\|(t))\int_0^t (t-s)^{-\frac{\alpha}{3}} \,\mathrm{d}s &\le C(\alpha)(\mathrm{essup}_{t\in[0,T]}\|\nabla h\|_{L_4}(t))^2 \\
&\le C(\alpha)(\mathrm{essup}_{t\in[0,T]}\|h\|_{H^2}(t))^2 \le C_1(h_0,T) < \infty.
\end{aligned}$$

If we combine above results, then we come to the following conclusion,

$$\|h\|_{L_\infty(0,T;H^{2\alpha})} \le C_{2\alpha}(h_0, T),$$

as desired.

Step 3. For $\alpha < 2$ we show

$$\|h\|_{L_\infty(0,T;H^{2\alpha})} \leq C_{2\alpha}(h_0, T) + Ct^{\frac{3-2\alpha}{6}}\|h_0\|_{H^3},$$

with the same method. We continue our calculations

$$\begin{aligned}\|h\|_{H^{2\alpha}} &\leq \|e^{\Delta^3 t}h_0\|_{H^{2\alpha}} + \frac{DK}{2}\int_0^t \|(Id-\Delta)^\alpha e^{\Delta^3(t-s)}|\nabla h(s,\cdot)|^2\| \, \mathrm{d}s \\ &\quad + K\int_0^t \|(Id-\Delta)^{\alpha+2}e^{\Delta^3(t-s)}h\| \, \mathrm{d}s + K\int_0^t \|(Id-\Delta)^{\alpha+1}e^{\Delta^3(t-s)}\mathrm{div}\, F\| \, \mathrm{d}s \\ &= \|e^{\Delta^3 t}h_0\|_{H^{2\alpha}} + I_1 + I_2 + I_4.\end{aligned}$$

Observe that

$$\|e^{\Delta^3 t}h_0\|_{H^{2\alpha}} \leq \|(1+|\cdot|^2)^{\alpha-\frac{3}{2}}e^{-|\cdot|^6 t}(1+|\cdot|^2)^{\frac{3}{2}}\hat{h}_0(\cdot)\| \leq C(\alpha)t^{-\alpha/3+1/2}.$$

Moreover,

$$\begin{aligned}|I_2| &\leq K\int_0^t \|(1+|\cdot|^2)^{\alpha+1}e^{-|\cdot|^6(t-s)}(1+|\cdot|^2)\hat{h}(s,\cdot)\| \, \mathrm{d}s \\ &\leq C(\alpha)\mathrm{essup}_{t\in[0,T]}\|h\|_{H^2}(t)\int_0^t (t-s)^{-\frac{\alpha+1}{3}} \, \mathrm{d}s \leq C_{2\alpha}(h_0, T),\end{aligned}$$

since $\alpha < 2$. Fix δ such that $\alpha < 2-\delta$. We then have

$$\begin{aligned}|I_3| &\leq \frac{K}{3}\int_0^t \|(1+|\cdot|^2)^{\alpha+1+\delta}e^{-|\cdot|^6(t-s)}(1+|\cdot|^2)^{1/2-\delta}\hat{h}_x^3(s,\cdot)\| \, \mathrm{d}s \\ &\leq C\mathrm{essup}_{t\in[0,T]}\|h_x^3\|_{H^{1-2\delta}}(t)\int_0^t (t-s)^{-\frac{\alpha+1+\delta}{3}} \, \mathrm{d}s \leq C(\alpha,\delta)\mathrm{essup}_{t\in[0,T]}\|h\|_{H^{3-2\delta}}(t).\end{aligned}$$

We estimate I_1 as before.

In particular, if $\alpha = \frac{3}{2}$ we obtain the desired result. □

Summing up, we can give a *proof of Theorem 1.* Namely, Theorem 3 yields local in time existence of weak solutions while the estimates provided by Theorem 4 imply global existence of solutions. Hence, it only remains to show uniqueness.

4 Uniqueness of the solutions

In this section we show that the weak solutions we constructed are indeed unique.

Theorem 5 *Let us assume that h is a weak solution to (1) with the initial condition (2), where $h_0 \in H^3$. Then, this is a unique solution.*

Proof. By Theorem 4, any weak solution will be in $L_\infty(0,T;H^3)$ provided that the initial condition is in H^3. Consider the equation for the difference, $h = h_1 - h_2$, where h_1 and h_2 are two weak solutions with the same initial condition. Testing this equation with h we arrive at the following identity,

$$\frac{1}{2}\frac{d}{dt}\|h\|^2 + \|\nabla\Delta h\|^2 = \|\Delta h\|^2 + \int_{\mathbb{T}^2}[\frac{D}{2}(|\nabla h_2|^2 - |\nabla h_1|^2)h + (F(h_1)-F(h_2))\nabla\Delta h]. \tag{18}$$

It is sufficient to estimate the nonlinear generic terms on the RHS. Let us look at

$$I = \int_{\mathbb{T}^2} ((h_{2,x}^3 - h_{1,x}^3)\Delta h_x = -\int_{\mathbb{T}^2} h_x(h_{2,x}^2 + h_{2,x}h_{1,x} + h_{1,x}^2)\Delta h_x.$$

The term in the parenthesis may be bounded by $3K^2$, where

$$K = \|h\|_{L_\infty(0,T;H^3)}.$$

Thus,

$$|I| \le \frac{9}{4\epsilon}K^4\|h_x\|^2 + \epsilon\|\Delta h_x\|^2$$

where ϵ shall be chosen later.

We may bound the remaining cubic and the quadratic terms in the same way. This yields the estimates,

$$\left|\int_{\mathbb{T}^2}(F(h_2) - F(h_1))\nabla\Delta h\right| \le \frac{C_3(K)}{\epsilon}\|\nabla h\|^2 + \epsilon(\frac{\alpha}{3} + \beta)\|\nabla\Delta h\|^2,$$
$$\left|\frac{D}{2}\int_{\mathbb{T}^2}[(|\nabla h_2|^2 - |\nabla h_1|^2)h\right| \le C_2(K)\frac{D}{2}\|\nabla h\|^2 + \frac{D}{4}\|h\|^2.$$

As a result we obtain:

$$\frac{1}{2}\frac{d}{dt}\|h\|^2 + \|\nabla\Delta h\|^2 \le \|\Delta h\|^2 + \frac{D}{4}\|h\|^2 + C_2(K)\frac{D}{2}\|\nabla h\|^2 + \frac{C_3(K)}{\epsilon}\|\nabla h\|^2 + \epsilon(\frac{\alpha}{3} + \beta)\|\nabla\Delta h\|^2. \quad (19)$$

We now choose ϵ so that $(\frac{\alpha}{3} + \beta)\epsilon = 1/2$.

In order to continue, we need the interpolation lemma below.

Lemma 4 *Let us suppose that $u \in H^3$, then for any $\epsilon > 0$ there is a constant $C_\epsilon > 0$ so that*

$$\|\Delta u\| \le C_\epsilon\|u\| + \epsilon\|\nabla\Delta u\|.$$

Proof. Let $C_\varepsilon = \sup_{x\in[0,\infty)} x^2 - \varepsilon x^3 < \infty$. Then,

$$\|\Delta u\| = \||\cdot|^2\hat{u}(\cdot)\| \le \|C_\varepsilon\hat{u}(\cdot) + \varepsilon|\cdot|^3\hat{u}(\cdot)\| \le C_\varepsilon\|u\| + \varepsilon\|\nabla\Delta u\|. \quad \square$$

Combining this Lemma with $\|\nabla h\| \le C(\mathbb{T}^2)\|\Delta u\|$ we conclude

$$\frac{1}{2}\frac{d}{dt}\|h\|^2 + \frac{1}{2}\|\nabla\Delta h\|^2 \le K_\epsilon\|h\|^2 + M\epsilon\|\nabla\Delta h\|^2.$$

We choose again ϵ, so that $M\epsilon = \frac{1}{2}$. We apply Gronwall inequality to the resulting estimate,

$$\frac{1}{2}\frac{d}{dt}\|h\|^2 \le K_\epsilon\|h\|^2.$$

Since $h(0) = 0$, we conclude that $h(t) = 0$ for all $t \in [0, T]$. Uniqueness follows. $\square$

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