

On the Brunn-Minkowski inequality for general measures with applications to new isoperimetric type-inequalities

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Abstract

We show that for any product measure $\mu = \mu_1 \otimes \dots \otimes \mu_n$ on \mathbb{R}^n , where μ_i have symmetric unimodal densities, the inequality

$$\mu(\lambda A + (1 - \lambda)B)^{1/n} \geq \lambda \mu(A)^{1/n} + (1 - \lambda) \mu(B)^{1/n}$$

holds true for any non-empty ideals $A, B \subseteq \mathbb{R}^n$. Moreover, using log-Brunn-Minkowski type inequalities due to C. Saroglou, [S2], and to D. Cordero-Erausquin, M. Fradelizi and B. Maurey, [CFM], we prove the above inequality for any ideals A, B and unconditional log-concave measures, as well as for general convex symmetric sets and even log-concave measures on the plane.

In addition, we deduce $\frac{1}{n}$ -concavity of the parallel volumes $t \mapsto \mu(A + tB)$, Brunn's type theorem and certain analogues of Minkowski first inequality. We also provide examples showing optimality of our assumptions.

As a consequence of the above result, the Gaussian Brunn-Minkowski inequality holds true in the case of unconditional convex set, as well as in the case of general symmetric sets on the plane. This partially solves the conjecture proposed by R. Gardner and the fourth named author in [GZ].

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1 Introduction

The classical Brunn-Minkowski inequality states that for any two non-empty compact sets A, B in \mathbb{R}^n and any $\lambda \in [0, 1]$ we have

$$\text{vol}_n(\lambda A + (1 - \lambda)B)^{1/n} \geq \lambda \text{vol}_n(A)^{1/n} + (1 - \lambda) \text{vol}_n(B)^{1/n}, \quad (1)$$

with equality if and only if $B = aA + b$, where $a > 0$ and $b \in \mathbb{R}^n$. Here vol_n stands for the Lebesgue measure on \mathbb{R}^n and

$$A + B = \{a + b : a \in A, b \in B\}$$

is the Minkowski sum of A and B . Due to homogeneity of the volume, this inequality is equivalent to $\text{vol}_n(A + B)^{1/n} \geq \text{vol}_n(A)^{1/n} + \text{vol}_n(B)^{1/n}$. The Brunn-Minkowski inequality turns out to be a powerful tool. In particular, it easily implies the classical isoperimetric inequality: for any compact set $A \subset \mathbb{R}^n$ we have $\text{vol}_n(A_t) \geq \text{vol}_n(B_t)$, where B is an Euclidean ball satisfying $\text{vol}_n(A) = \text{vol}_n(B)$ and A_t stands for the t -enlargement of A , i.e., $A_t = A + tB_2^n$, where B_2^n is the unit Euclidean ball, $B_2^n = \{x : |x| = 1\}$. To see this it is enough to observe that

$$\text{vol}_n(A + tB_2^n)^{1/n} \geq \text{vol}_n(A)^{1/n} + \text{vol}_n(tB_2^n)^{1/n} = \text{vol}_n(B)^{1/n} + \text{vol}_n(tB_2^n)^{1/n} = \text{vol}_n(B + tB_2^n)^{1/n}.$$

Taking $t \rightarrow 0^+$ one gets a more familiar form of isoperimetry: among all sets with fixed volume the surface area

$$\text{vol}_n^+(\partial A) = \liminf_{t \rightarrow 0^+} \frac{\text{vol}_n(A + tB_2^n) - \text{vol}_n(A)}{t}$$

is minimized in the case of the Euclidean ball. We refer to [G] for more information on Brunn-Minkowski-type inequalities.

Using the inequality between means one gets an a priori weaker dimension free form of (1), namely

$$\text{vol}_n(\lambda A + (1 - \lambda)B) \geq \text{vol}_n(A)^\lambda \text{vol}_n(B)^{1-\lambda}. \quad (2)$$

In fact (2) and (1) are equivalent. To see this one has to take $\tilde{A} = A/\text{vol}_n(A)^{1/n}$, $\tilde{B} = B/\text{vol}_n(B)^{1/n}$ and $\lambda = \lambda \text{vol}_n(A)^{1/n}/(\lambda \text{vol}_n(A)^{1/n} + (1 - \lambda) \text{vol}_n(B)^{1/n})$ in (2). This phenomenon is a consequence of homogeneity of the Lebesgue measure.

The above notions can be generalized to the case of the so-called s -concave measures. Here we assume that $s \in (0, 1]$, whereas in general the notion of s -concave measures makes sense for any $s \in [-\infty, \infty]$. We say that a measure μ on \mathbb{R}^n is s -concave if for any non-empty compact sets $A, B \subset \mathbb{R}^n$ we have

$$\mu(\lambda A + (1 - \lambda)B)^s \geq \lambda \mu(A)^s + (1 - \lambda) \mu(B)^s. \quad (3)$$

Similarly, a measure μ is called log-concave (or 0-concave) if for any compact $A, B \subset \mathbb{R}^n$ we have

$$\mu(\lambda A + (1 - \lambda)B) \geq \mu(A)^\lambda \mu(B)^{1-\lambda}. \quad (4)$$

Let us assume that the support of our measure is non degenerate, i.e., is not contained in any affine subspace of \mathbb{R}^n of dimension less than n . It turns out that, in this case, there is a very useful description of log-concave and s -concave measures due to Borell, see [B]: a measure μ is log-concave if and only if it has a density of the form $\varphi = e^{-V}$, where V is convex (and may attain value $+\infty$). Such functions are called log-concave. Moreover, μ is s -concave with $s \in (0, 1/n)$ if and only if it has a density φ such that $\varphi^{1-s/n}$ is concave. In the case $s = 1/n$ the density has to satisfy the strongest condition $\varphi(\lambda x + (1 - \lambda)y) \geq \max(\varphi(x), \varphi(y))$. An example of such measure is the uniform measure on a convex body $K \subset \mathbb{R}^n$. Let us also notice that a measure with non-degenerate support cannot be s -concave with $s > \frac{1}{n}$. It can be seen by taking $\tilde{A} = \varepsilon A$, $\tilde{B} = \varepsilon B$, sending $\varepsilon \rightarrow 0^+$ and comparing the limit with the Lebesgue measure.

Inequality (2) says that Lebesgue measure is log-concave, whereas (1) means that it is also $1/n$ -concave. In general log-concavity does not imply s -concavity for $s > 0$. Indeed, consider the standard Gaussian measure γ_n on \mathbb{R}^n , i.e., a measure with density $(2\pi)^{-n/2} \exp(-|x|^2/2)$. This density is clearly log-concave and therefore γ_n satisfies (4). To see that γ_n does not satisfy (3) for $s > 0$ it suffices to take $B = \{x\}$ and send $x \rightarrow \infty$. Then the left hand side converges

to 0 while the right hand side stays equal to $\lambda\mu(A)^s$, which is strictly positive for $\lambda > 0$ and $\mu(A) > 0$.

One might therefore ask whether (3) holds true for γ_n if we restrict ourselves to some special class of subsets of \mathbb{R}^n . In [GZ] R. Gardner and the fourth named author conjectured (Question 7.1) that

$$\gamma_n(\lambda A + (1 - \lambda)B)^{1/n} \geq \lambda\gamma_n(A)^{1/n} + (1 - \lambda)\gamma_n(B)^{1/n} \quad (5)$$

holds true for any closed convex sets with $0 \in A \cap B$ and $\lambda \in [0, 1]$ and verified this conjecture in the following cases:

- (a) when A and B are products of intervals containing the origin,
- (b) when $A = [-a_1, a_2] \times \mathbb{R}^{n-1}$, where $a_1, a_2 > 0$ and B is arbitrary,
- (c) when $A = aK$ and $B = bK$ where $a, b > 0$ and K is a convex set, symmetric with respect to the origin.

It is interesting to note that the case (c) is related to the so-called Banaszczyk's theorem (B-theorem), [CFM]. It states that for any convex symmetric set K the function $t \mapsto \gamma_n(e^t K)$ is log-concave. Moreover, the same is true for any unconditional log-concave measures and unconditional sets. It is an open question (called B-conjecture) whether this property is satisfied for any centrally symmetric log-concave measures and centrally symmetric convex sets. This question has an affirmative answer for $n = 2$ due to the works of Livne Bar-on [Li] and of Saroglou [S2]. In [S2] the proof is done by linking the problem to the new logarithmic-Brunn-Minkowski inequality of Böröczky, Lutwak, Yang and Zhang, see [BLYZ1], [BLYZ2], [S1] and [S2].

It turns out that the assertion of the B -conjecture for a measure μ and a symmetric convex body K formally implies the inequality

$$\mu(\lambda aK + (1 - \lambda)bK)^{1/n} \geq \lambda\mu(aK)^{1/n} + (1 - \lambda)\mu(bK)^{1/n},$$

i.e., the Brunn-Minkowski inequality is satisfied for dilations of K , see [M2, Proposition 3.1].

In [NT] T. Tkocz and the third named author showed that in general (5) is false under the assumption $0 \in A \cap B$. For sufficiently small $\varepsilon > 0$ and $\alpha < \pi/2$ sufficiently close to $\pi/2$ the pair of sets

$$A = \{(x, y) \in \mathbb{R}^2 : y \geq |x| \tan \alpha\}, \quad B = \{(x, y) \in \mathbb{R}^2 : y \geq |x| \tan \alpha - \varepsilon\}$$

serves as a counterexample. The authors however conjectured that (5) should be true for (centrally) symmetric convex bodies A, B .

Through this note \mathcal{K} is some family of sets closed under dilations, i.e., $A \in \mathcal{K}$ implies $tA \in \mathcal{K}$ for any $t > 0$. In particular, we assume that for any $K \in \mathcal{K}$ we have $0 \in K$. A general form of the Brunn-Minkowski inequality can be stated as follows.

Definition 1. We say that a Borel measure μ on \mathbb{R}^n satisfies the Brunn-Minkowski inequality in the class of sets \mathcal{K} if for any $A, B \in \mathcal{K}$ and for any $\lambda \in [0, 1]$ we have

$$\mu(\lambda A + (1 - \lambda)B)^{1/n} \geq \lambda\mu(A)^{1/n} + (1 - \lambda)\mu(B)^{1/n}. \quad (6)$$

Before we state our results, we introduce some basic notation and definitions.

Definition 2.

1. We say that a function $f : \mathbb{R}^n \rightarrow [0, \infty)$ is unconditional if it satisfies the following two properties.

(i) For any choice of signs $\varepsilon_1, \dots, \varepsilon_n \in \{-1, 1\}$ and any $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ we have $f(\varepsilon_1 x_1, \dots, \varepsilon_n x_n) = f(x)$.

(ii) For any $1 \leq i \leq n$ and any $x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n \in \mathbb{R}$ the function

$$t \mapsto f(x_1, \dots, x_{i-1}, t, x_{i+1}, \dots, x_n)$$

is non-increasing on $[0, \infty)$.

2. A set $A \subseteq \mathbb{R}^n$ (which is not necessarily a product set) is called an *ideal* if $\mathbf{1}_A$ is unconditional. In other words, a set $A \subseteq \mathbb{R}^n$ is called unconditional if $(x_1, \dots, x_n) \in A$ implies $(\varepsilon_1 x_1, \dots, \varepsilon_n x_n) \in A$ for any choice of signs $\varepsilon_1, \dots, \varepsilon_n \in \{-1, 1\}$. The class of all ideal (in \mathbb{R}^n) will be denoted by \mathcal{K}_I .

3. A set $A \subseteq \mathbb{R}^n$ is called symmetric if $A = -A$. The class of all symmetric convex sets in \mathbb{R}^n will be denoted by \mathcal{K}_S .

4. A measure μ on \mathbb{R}^n is called unconditional if it has an unconditional density. Note that if an unconditional measure μ is a product measure, i.e. $\mu = \mu_1 \otimes \dots \otimes \mu_n$, then the measures μ_i are unconditional on \mathbb{R} .

Our first theorem reads as follows.

Theorem 1. Let μ be an unconditional product measure on \mathbb{R}^n . Then μ satisfies the Brunn-Minkowski inequality in the class \mathcal{K}_I of all ideal in \mathbb{R}^n .

As we already mentioned, in such inequality for the measure with non-degenerate support, the power $1/n$ is the best possible. In addition, it will be shown in Examples 1 and 2, in the end of the paper, that neither the assumption that μ is a product measure, nor the unconditionality of our sets A and B can be dropped.

Our strategy is to prove a certain functional version of (6). A functional version of the classical Brunn-Minkowski inequality is called the Prekopa-Leindler inequality, see [G] for the proof.

Theorem 2 (Prekopa-Leindler inequality, [P], [Le]). Let f, g, m be non-negative measurable functions on \mathbb{R}^n and let $\lambda \in [0, 1]$. If for all $x, y \in \mathbb{R}^n$ we have $m(\lambda x + (1 - \lambda)y) \geq f(x)^\lambda g(y)^{1-\lambda}$ then

$$\int m \, dx \geq \left(\int f \, dx \right)^\lambda \left(\int g \, dx \right)^{1-\lambda}.$$

Here we prove an analogue of the above inequality for unconditional product measures and unconditional functions f, g, m .

Proposition 1. Fix $\lambda, p \in (0, 1)$. Suppose that m, f, g are unconditional. Let μ be an unconditional product measure on \mathbb{R}^n . Assume that for any $x, y \in \mathbb{R}^n$ we have

$$m(\lambda x + (1 - \lambda)y) \geq f(x)^p g(y)^{1-p}.$$

Then

$$\int m \, d\mu \geq \left[\left(\frac{\lambda}{p} \right)^p \left(\frac{1 - \lambda}{1 - p} \right)^{1-p} \right]^n \left(\int f \, d\mu \right)^p \left(\int g \, d\mu \right)^{1-p}.$$

The above proposition allows us to prove the following lemma, which is in fact a reformulation of Theorem 1.

Lemma 1. Let A, B be ideals in \mathbb{R}^n and let μ be an unconditional product measure on \mathbb{R}^n . Then for any $\lambda \in [0, 1]$ and $p \in (0, 1)$ we have

$$\mu(\lambda A + (1 - \lambda)B) \geq \left[\left(\frac{\lambda}{p} \right)^p \left(\frac{1 - \lambda}{1 - p} \right)^{1-p} \right]^n \mu(A)^p \mu(B)^{1-p}.$$

It is worth noticing that the factor on the right hand side of this inequality replaces is some sense the lack of homogeneity of our measure μ . The main idea of the proof is to introduce an additional parameter $p \neq \lambda$ and do the optimization with respect to p .

In the second part of this article we provide a link between the Brunn-Minkowski inequality and the logarithmic-Brunn-Minkowski inequality. To state our observation we need two definitions.

Definition 3. Let \mathcal{K} be a class of subsets closed under dilations. We say that a family $\odot = (\odot_\lambda)_{\lambda \in [0, 1]}$ of functions acting on $\mathcal{K} \times \mathcal{K}$ (and having values in the class of measurable subsets of \mathbb{R}^n) is a geometric mean if for any $A, B \in \mathcal{K}$ the set $A \odot_\lambda B$ is measurable, satisfies an inclusion $A \odot_\lambda B \subseteq \lambda A + (1 - \lambda)B$, and for any $s, t > 0$ there is $(sA) \odot_\lambda (tB) = s^\lambda t^{1-\lambda} (A \odot_\lambda B)$.

Definition 4. We say that a Borel measure μ on \mathbb{R}^n satisfies the logarithmic-Brunn-Minkowski inequality in the class of sets \mathcal{K} with a geometric mean \odot , if for any sets $A, B \in \mathcal{K}$ and for any $\lambda \in [0, 1]$ we have

$$\mu(A \odot_\lambda B) \geq \mu(A)^\lambda \mu(B)^{1-\lambda},$$

Remark 1. In what follows we use two different geometric means. First one is a geometric mean $\odot^S : \mathcal{K}_s \times \mathcal{K}_s \rightarrow \mathcal{K}_s$, defined by the formula

$$A \odot_\lambda^S B = \{x \in \mathbb{R}^n : \langle x, u \rangle \leq h_A^\lambda(u) h_B^{1-\lambda}(u), \forall u \in S^{n-1}\}.$$

Here h_A is a support function of A , i.e., $h_A(u) = \sup_{x \in A} \langle x, u \rangle$.

The second mean $\odot^I : \mathcal{K}_I \times \mathcal{K}_I \rightarrow \mathcal{K}_I$ is defined by

$$A \odot_\lambda^I B = \bigcup_{x \in A, y \in B} [-|x_1|^\lambda |y_1|^{1-\lambda}, |x_1|^\lambda |y_1|^{1-\lambda}] \times \dots \times [-|x_n|^\lambda |y_n|^{1-\lambda}, |x_n|^\lambda |y_n|^{1-\lambda}].$$

It is straightforward to check, with the help of the inequality $a^\lambda b^{1-\lambda} \leq \lambda a + (1 - \lambda)b$, $a, b \geq 0$, that both means are indeed geometric.

In the Section 3 we prove the following fact.

Proposition 2. Suppose that a Borel measure μ with a radially decreasing density f , i.e. density satisfying $f(tx) \geq f(x)$ for any $x \in \mathbb{R}^n$ and $t \in [0, 1]$, satisfies the logarithmic-Brunn-Minkowski inequality, with a geometric mean \odot , in a certain class of sets \mathcal{K} . Then μ satisfies the Brunn-Minkowski inequality in the class \mathcal{K} .

In [S2] C. Saroglou proved the logarithmic-Brunn-Minkowski inequality with geometric mean \odot^S for measures with even log-concave densities (that is, densities of the form e^{-V} , where V is even, i.e. $V(x) = V(-x)$, and convex) in the plane \mathbb{R}^2 , in the class of symmetric convex sets (see Corollary 3.3 therein). Thus, as a consequence of Proposition 2 and Remark 1, we get the following corollary.

Corollary 1. Let μ be a measure on \mathbb{R}^2 with an even log-concave density. Then μ satisfies the Brunn-Minkowski inequality in the class \mathcal{K}_s of all symmetric convex sets in \mathbb{R}^2 .

Moreover, in [CFM] (Proposition 8) the authors proved the following theorem (see also Proposition 4.2 in [S1]).

Theorem 3. The logarithmic-Brunn-Minkowski inequality holds true with the geometric mean \odot^I for any measure with unconditional log-concave density in the class \mathcal{K}_I of ideal in \mathbb{R}^n

We recall the argument in Section 3. As a consequence, applying our Proposition 2 together with Remark 1, we deduce the following fact.

Corollary 2. Let μ be an unconditional log-concave measure on \mathbb{R}^n . Then μ satisfies the Brunn-Minkowski inequality in the class \mathcal{K}_I of all ideals in \mathbb{R}^n .

Let us now briefly describe some corollaries of the Brunn-Minkowski type inequality we established, which are analogues to well-known offsprings of Brunn-Minkowski inequality for the volume. In what follows a pair (\mathcal{K}, μ) is called *nice* if one of the following three cases holds.

- (a) $\mathcal{K} = \mathcal{K}_I$ and μ is an unconditional product measure on \mathbb{R}^n
- (b) $\mathcal{K} = \mathcal{K}_I$ and μ is an unconditional log-concave measure on \mathbb{R}^n
- (c) $\mathcal{K} = \mathcal{K}_s$ and μ is an even log-concave measure on \mathbb{R}^2

Corollary 3. Suppose that a pair (\mathcal{K}, μ) is nice. Let $A, B \subset \mathcal{K}$ be convex. Then the function $t \mapsto \mu(A + tB)^{1/n}$ is concave on $[0, \infty)$.

Indeed, for any $\lambda \in [0, 1]$ and $t_1, t_2 \geq 0$ we have

$$\begin{aligned} \mu(A + (\lambda t_1 + (1 - \lambda)t_2)B)^{1/n} &= \mu((\lambda(A + t_1B) + (1 - \lambda)(A + t_2B))^{1/n}) \\ &\geq \lambda\mu(A + t_1B)^{1/n} + (1 - \lambda)\mu(A + t_2B)^{1/n}. \end{aligned}$$

Note that in the first line we have used the convexity of A and B . If $B = B_2^n$ is the Euclidean ball, the expression $\mu(A + tB)$ is called the parallel volume and has been studied in the case of Lebesgue measure by Costa and Cover in [C] as an analogue of concavity of entropy power in information theory. The authors conjectured that for any measurable set A the parallel volume is $\frac{1}{n}$ -concave. In [FM] it has been shown that in general this conjecture is false. The authors

proved the conjecture on the real line. They also showed that it holds true on the plane in the case of connected sets. In a recent paper [M] the second named author investigated the parallel volumes $\mu(A + tB_2^n)$ in the context of s -concave measures. He proved that $t \mapsto \mu(A + t[-1, 1])^{1/s}$ is concave for s -concave measures on the real line. However, as we already mentioned, even for the Lebesgue measure this result cannot be generalized to the higher dimensions. Our Corollary 3 gives the Costa-Cover conjecture for any convex set $A \in \mathcal{K}$, where (\mathcal{K}, μ) is a nice pair. Moreover, the Euclidean ball B_2^n can be replaced with any convex set $B \in \mathcal{K}$.

Second, we may state the following analogue of the Brunn's theorem on volumes of sections of convex bodies.

Corollary 4. Suppose that a pair (\mathcal{K}, μ) is nice. Let $K \in \mathcal{K}$ be a convex set. Then the function $t \mapsto \mu(A \cap \{x_1 = t\})$ is $\frac{1}{n-1}$ -concave on its support.

To prove this let us take $K_t = K \cap \{x_1 = t\}$. By convexity of K we get $\lambda K_{t_1} + (1 - \lambda)K_{t_2} \subseteq K_{\lambda t_1 + (1-\lambda)t_2}$. Thus, using (6), for any $\lambda \in [0, 1]$ and $t_1, t_2 \in \mathbb{R}$ such that K_{t_1} and K_{t_2} are both non-empty, we get

$$\mu(K_{\lambda t_1 + (1-\lambda)t_2})^{\frac{1}{n-1}} \geq \mu(\lambda K_{t_1} + (1 - \lambda)K_{t_2})^{\frac{1}{n-1}} \geq \lambda \mu(K_{t_1})^{\frac{1}{n-1}} + (1 - \lambda) \mu(K_{t_2})^{\frac{1}{n-1}}.$$

Third, let us mention the relation of our result to the Gaussian isoperimetric inequality, Ehrhard's inequality and the so-called S-inequality. Let us begin with the Gaussian isoperimetry. It turns out that for any measurable set $A \subset \mathbb{R}^n$ and any $t > 0$, the quantity $\gamma_n(A_t)$ is minimized, among all sets with prescribed measure, for the half spaces $H_{a,\theta} = \{x \in \mathbb{R}^n : \langle x, \theta \rangle \leq a\}$, with $a \in \mathbb{R}$ and $\theta \in S^{n-1}$. This is a famous result established by Sudakov and Tsirelson, [ST], as well as independently by Borell, [B2], and can be seen as one of the cornerstones of the concentration of measure theory. Infinitesimally, it says that among all sets with prescribed measure the half spaces are those with the smallest Gaussian measure of the boundary, i.e., the quantity

$$\gamma_n^+(A) = \liminf_{t \rightarrow 0^+} \frac{\gamma_n(A + tB_2^n) - \gamma_n(A)}{t}.$$

A more general statement is the so-called Ehrhard's inequality. It says that for any non-empty measurable sets A, B we have

$$\Phi^{-1}(\gamma_n(\lambda A + (1 - \lambda)B)) \geq \lambda \Phi^{-1}(\gamma_n(A)) + (1 - \lambda) \Phi^{-1}(\gamma_n(B)), \quad \lambda \in [0, 1], \quad (7)$$

where $\Phi(t) = \gamma_1((-\infty, a])$. This inequality has been considered for the first time by Ehrhard in [E], where the author proved it assuming that both A and B are convex. Then Latała in [L] generalized Ehrhard's result to the case of arbitrary A and convex B . In it's full generality, the inequality (7) has been established by Borell, [B3]. Note that our inequality (5), valid for unconditional ideals, is an inequality of the same type, with $\Phi(t)$ replaced with t^n . None of them is a direct consequence of the other. However, unlike the Gaussian Brunn-Minkowski inequality, the Ehrhard's inequality (in fact a more general form of it where λ and $1 - \lambda$ are replaced with α and β , under the conditions $\alpha + \beta \geq 1$ and $|\alpha - \beta| \leq 1$) gives the Gaussian isoperimetry as a simple consequence.

Let us also describe the S-inequality of Latała and Oleszkiewicz, see [LO]. It states that for any $t > 1$ and any symmetric convex body K the quantity $\gamma_n(tK)$ is minimized, among all subsets with prescribed measure, for the strips of the form $S_L = \{x \in \mathbb{R}^n : |x_1| \leq L\}$. This

result admits an equivalent infinitesimal version, namely, among all symmetric convex bodies K with prescribed Gaussian measure the strips S_L minimize the quantity $\frac{d}{dt}\gamma_n(tK)|_{t=1}$, which is equivalent to maximizing

$$M(K) = \int_K |x|^2 d\gamma_n(x),$$

see [KS] or [NT3]. For a general measure μ with a density $e^{-\psi}$, one can show that the infinitesimal version of S-inequality is an issue of maximizing the quantity

$$M_\mu(K) = - \int_K \langle x, \nabla\psi \rangle d\mu(x), \quad (8)$$

see equation (11) below. Not much is known about this general problem. In the unconditional case it has been solved for some particular product measures like products of Gamma and Weibull distributions, see [NT2]. It turns out that inequality (5) implies a certain mixture of Gaussian isoperimetry and reverse S-inequality. Namely, we have the following Corollary.

Corollary 5. Let A, B be two ideals in \mathbb{R}^n (or general symmetric convex sets in \mathbb{R}^2) and let $r > 0$. Then we have

$$r\gamma_n^+(A) + M(A) \geq n\gamma_n(rB_2^n)^{\frac{1}{n}}\gamma_n(A)^{1-\frac{1}{n}}$$

with equality for $A = rB_2^n$.

Let us state and prove a more general version of Corollary 5. Let $\mu^+(A)$ be the μ boundary measure of A , i.e.,

$$\mu^+(A) = \liminf_{t \rightarrow 0^+} \frac{\mu(A + tB_2^n) - \mu(A)}{t}.$$

Moreover, let us introduce the so-called first mixed volume of arbitrary sets A and B , with respect to measure μ . Namely

$$V_1^\mu(A, B) = \frac{1}{n} \liminf_{t \rightarrow 0^+} \frac{\mu(A + tB) - \mu(A)}{t}.$$

Clearly, $\mu^+(A) = nV_1^\mu(A, B_2^n)$.

Corollary 6. Let $A, B \in \mathcal{K}$ and suppose that (\mathcal{K}, μ) is a nice pair. Then we have

$$V_1^\mu(A, B) + \frac{1}{n}M_\mu(A, B) \geq \mu(B)^{1/n}\mu(A)^{1-1/n}. \quad (9)$$

In particular,

$$r\mu^+(\partial A) + M_\mu(A) \geq n\mu(rB_2^n)^{1/n}\mu(A)^{1-1/n}. \quad (10)$$

To prove this we note that for any sets $A, B \in \mathcal{K}$ and any $\varepsilon \in [0, 1)$ we have

$$\mu(A + \varepsilon B)^{1/n} \geq (1 - \varepsilon)\mu\left(\frac{A}{1 - \varepsilon}\right)^{1/n} + \varepsilon\mu(B)^{1/n}.$$

Indeed, it suffices to use Theorem 1 with $\lambda = 1 - \varepsilon$ and $\tilde{A} = A/(1 - \varepsilon)$, $\tilde{B} = B$. Note that for $\varepsilon = 0$ we have equality. Thus, differentiating at $\varepsilon = 0$ we get

$$\frac{1}{n}\mu(A)^{\frac{1}{n}-1} \cdot nV_1^\mu(A, B) \geq \mu(B)^{\frac{1}{n}} - \mu(A)^{\frac{1}{n}} + \frac{1}{n}\mu(A)^{\frac{1}{n}-1} \frac{d}{dt}\mu(tA)\Big|_{t=1}.$$

By changing variables we obtain

$$\left. \frac{d}{dt} \mu(tA) \right|_{t=1} = \left. \frac{d}{dt} \int_A e^{-\psi(tx)} t^n dx \right|_{t=1} = n\mu(A) - \int_A \langle x, \nabla \psi(x) \rangle d\mu(x) = n\mu(A) - M_\mu(A). \quad (11)$$

Thus,

$$\mu(A)^{\frac{1}{n}-1} V_1^\mu(A, B) \geq \mu(B)^{\frac{1}{n}} - \frac{1}{n} \mu(A)^{\frac{1}{n}-1} M_\mu(A).$$

This is exactly (9). To get (10) one has to take $B = rB_2^n$ in (9).

The above inequalities can be seen as an analogues of the so-called Minkowski first inequality for the Lebesgue measure, see [G] and [Sn], which says that for any two convex bodies A, B in \mathbb{R}^n we have

$$V_1^{\text{vol}_n}(A, B) \geq \text{vol}_n(A)^{1-\frac{1}{n}} \text{vol}_n(B)^{\frac{1}{n}}.$$

The rest of this article is organized as follows. In the next section we first show how Lemma 1 implies Theorem 1. Then Lemma 1 is deduced from Proposition 1. Finally, we prove Proposition 1. In Section 3 we prove Proposition 2 and recall the proof of Theorem 3. In the last section we give examples showing optimality of some of our results and state open questions.

2 Proof of Theorem 1

We first show how Lemma 1 implies Theorem 1.

Proof of Theorem 1. Without loss of generality we can assume that $\lambda \in (0, 1)$. Let us assume for a moment that $\mu(A)\mu(B) > 0$. Then we can use Lemma 1 with

$$p = \frac{\lambda\mu(A)^{1/n}}{\lambda\mu(A)^{1/n} + (1-\lambda)\mu(B)^{1/n}} \in (0, 1). \quad (12)$$

Note that

$$\frac{\lambda}{p} = \frac{\lambda\mu(A)^{1/n} + (1-\lambda)\mu(B)^{1/n}}{\mu(A)^{1/n}}, \quad \frac{1-\lambda}{1-p} = \frac{\lambda\mu(A)^{1/n} + (1-\lambda)\mu(B)^{1/n}}{\mu(B)^{1/n}}.$$

Then

$$\left[\left(\frac{\lambda}{p} \right)^p \left(\frac{1-\lambda}{1-p} \right)^{1-p} \right]^n \mu(A)^p \mu(B)^{1-p} = (\lambda\mu(A)^{1/n} + (1-\lambda)\mu(B)^{1/n})^n.$$

Thus the inequality in Lemma 1 becomes

$$\mu(\lambda A + (1-\lambda)B) \geq (\lambda\mu(A)^{1/n} + (1-\lambda)\mu(B)^{1/n})^n.$$

Now suppose that, say, $\mu(B) = 0$. Since B is a non-empty ideal, we have $0 \in B$. Therefore, $\lambda A \subseteq \lambda A + (1-\lambda)B$. Let φ_μ be the unconditional density of μ . Hence,

$$\begin{aligned} \mu(\lambda A + (1-\lambda)B) &\geq \mu(\lambda A) = \int_{\lambda A} \varphi(x) dx = \lambda^n \int_A \varphi(\lambda y) dy \\ &= \lambda^n \int_A \varphi(\lambda y_1, \dots, \lambda y_n) dy = \lambda^n \int_A \varphi(\lambda|y_1|, \dots, \lambda|y_n|) dy \\ &\geq \lambda^n \int_A \varphi(|y_1|, \dots, |y_n|) dy = \lambda^n \mu(A). \end{aligned}$$

Therefore,

$$\mu(\lambda A + (1 - \lambda)B)^{1/n} \geq \lambda\mu(A)^{1/n} = \lambda\mu(A)^{1/n} + (1 - \lambda)\mu(B)^{1/n}.$$

□

Next we show that Proposition 1 implies Lemma 1.

Proof of Lemma 1. We can assume that $\lambda \in (0, 1)$. Let us take $m(x) = \mathbf{1}_{\lambda A + (1 - \lambda)B}(x)$, $f(x) = \mathbf{1}_A(x)$, $g(x) = \mathbf{1}_B(x)$. Clearly, f, g and m are unconditional. It is easy to verify that for any $p \in (0, 1)$ we have $m(\lambda x + (1 - \lambda)y) \geq f(x)^p g(y)^{1-p}$. Our assertion follows from Proposition 1. □

For the proof of Proposition 1 we need a one dimensional Brunn-Minkowski inequality for unconditional measures.

Lemma 2. Let A, B be two symmetric intervals and let μ be unconditional on \mathbb{R} . Then for any $\lambda \in [0, 1]$ we have

$$\mu(\lambda A + (1 - \lambda)B) \geq \lambda\mu(A) + (1 - \lambda)\mu(B).$$

Proof. We can assume that $A = [-a, a]$ and $B = [-b, b]$ for some $a, b > 0$. Let φ_μ be the density of μ . Then our assertion is equivalent to

$$\int_0^{\lambda a + (1 - \lambda)b} \varphi_\mu(x) dx \geq \lambda \int_0^a \varphi_\mu(x) dx + (1 - \lambda) \int_0^b \varphi_\mu(x) dx.$$

In other words, the function $t \mapsto \int_0^t \varphi_\mu(x) dx$ should be concave on $[0, \infty)$. This is equivalent to $t \mapsto \varphi_\mu(t)$ being non-increasing on $[0, \infty)$. □

Proof of Proposition 1. We proceed by induction on n . Let us begin with the case $n = 1$. We can assume that $\|f\|_\infty, \|g\|_\infty > 0$. Indeed, if we multiply the functions m, f, g by numbers c_m, c_f, c_g satisfying $c_m = c_f^p c_g^{1-p}$, the hypothesis and the assertion do not change. Therefore, taking $c_f = \|f\|_\infty^{-1}$, $c_g = \|g\|_\infty^{-1}$, $c_m = \|f\|_\infty^{-p} \|g\|_\infty^{-(1-p)}$ we can assume that $\|f\|_\infty = \|g\|_\infty = 1$. Then the sets $\{f > t\}$ and $\{g > t\}$ are non-empty for $t \in (0, 1)$. Moreover, $\lambda\{f > t\} + (1 - \lambda)\{g > t\} \subseteq \{m > t\}$. Indeed, if $x \in \{f > t\}$ and $y \in \{g > t\}$ then $m(\lambda x + (1 - \lambda)y) \geq f(x)^p g(y)^{1-p} > t^p t^{1-p} = t$. Thus, $\lambda x + (1 - \lambda)y \in \{m > t\}$. Therefore, using Lemma 2, we get

$$\begin{aligned} \int m d\mu &= \int_0^\infty \mu(\{m > t\}) dt \geq \int_0^1 \mu(\{m > t\}) dt \\ &\geq \int_0^1 \mu(\lambda\{f > t\} + (1 - \lambda)\{g > t\}) dt \\ &\geq \lambda \int_0^1 \mu(\{f > t\}) dt + (1 - \lambda) \int_0^1 \mu(\{g > t\}) dt \\ &= \lambda \int_0^\infty \mu(\{f > t\}) dt + (1 - \lambda) \int_0^\infty \mu(\{g > t\}) dt \\ &= \lambda \int f d\mu + (1 - \lambda) \int g d\mu. \end{aligned}$$

Now, using the inequality $pa + (1-p)b \geq a^p b^{1-p}$ we get

$$\lambda \int f \, d\mu + (1-\lambda) \int g \, d\mu = p \frac{\lambda}{p} \int f \, d\mu + (1-p) \frac{1-\lambda}{1-p} \int g \, d\mu \quad (13)$$

$$\geq \left(\frac{\lambda}{p}\right)^p \left(\frac{1-\lambda}{1-p}\right)^{1-p} \left(\int f \, d\mu\right)^p \left(\int g \, d\mu\right)^{1-p}. \quad (14)$$

Next, we do the induction step. Let us assume that the assertion is true in dimension $n-1$. Let $m, f, g : \mathbb{R}^n \rightarrow [0, \infty)$ be unconditional. For $x_0, y_0, z_0 \in \mathbb{R}$ we define functions $m_{z_0}, f_{x_0}, g_{y_0}$ by

$$m_{z_0}(x) = m(z_0, x), \quad f_{x_0}(x) = f(x_0, x), \quad g_{y_0}(x) = g(y_0, x).$$

Clearly, these functions are also unconditional. Moreover, due to our assumptions on m, f, g we have

$$\begin{aligned} m_{\lambda x_0 + (1-\lambda)y_0}(\lambda x + (1-\lambda)y) &= m(\lambda x_0 + (1-\lambda)y_0, \lambda x + (1-\lambda)y) \\ &\geq f(x_0, x)^p g(y_0, y)^{1-p} = f_{x_0}(x)^p g_{y_0}(y)^{1-p}. \end{aligned}$$

Let us decompose μ in the form $\mu = \mu_1 \times \bar{\mu}$, where μ_1 is a measure on \mathbb{R} . Note that μ_1 and $\bar{\mu}$ are unconditional and $\bar{\mu}$ is a product measure. Thus, by our induction assumption we have

$$\int m_{\lambda x_0 + (1-\lambda)y_0} \, d\bar{\mu} \geq \left[\left(\frac{\lambda}{p}\right)^p \left(\frac{1-\lambda}{1-p}\right)^{1-p} \right]^{n-1} \left(\int f_{x_0} \, d\bar{\mu}\right)^p \left(\int g_{y_0} \, d\bar{\mu}\right)^{1-p}. \quad (15)$$

Now we define the functions

$$M(z_0) = \left[\left(\frac{\lambda}{p}\right)^p \left(\frac{1-\lambda}{1-p}\right)^{1-p} \right]^{-(n-1)} \int m_{z_0}(\xi) \, d\bar{\mu}(\xi), \quad (16)$$

$$F(x_0) = \int f_{x_0}(\xi) \, d\bar{\mu}(\xi), \quad G(y_0) = \int g_{y_0}(\xi) \, d\bar{\mu}(\xi). \quad (17)$$

Using nequality (15) we immediately get that

$$M(\lambda x_0 + (1-\lambda)y_0) \geq F(x_0)^p G(y_0)^{1-p}.$$

Moreover, it is easy to see that M, F, G are unconditional on \mathbb{R} . Thus, using Lemma 2 (the one-dimensional case), we get

$$\int M(z) \, d\mu_1(z) \geq \left(\frac{\lambda}{p}\right)^p \left(\frac{1-\lambda}{1-p}\right)^{1-p} \left(\int F(x) \, d\mu_1(x)\right)^p \left(\int G(y) \, d\mu_1(y)\right)^{1-p}. \quad (18)$$

Observe that

$$\begin{aligned} \int M(z) \, d\mu_1(z) &= \left[\left(\frac{\lambda}{p}\right)^p \left(\frac{1-\lambda}{1-p}\right)^{1-p} \right]^{-(n-1)} \int \int m_{z_0}(\xi) \, d\mu_{n-1}(\xi) \, d\mu_1(z_0) \\ &= \left[\left(\frac{\lambda}{p}\right)^p \left(\frac{1-\lambda}{1-p}\right)^{1-p} \right]^{-(n-1)} \int m \, d\mu. \end{aligned}$$

Similarly,

$$\int F(x_0) \, d\mu_1(x_0) = \int f \, d\mu, \quad \int G(y_0) \, d\mu_1(y_0) = \int g \, d\mu.$$

Our assertion follows. □

3 Proof of Proposition 2

In this section we first prove Proposition 2. The argument has a flavour of our previous proof.

Proof of Proposition 2. Let us first assume that $\mu(A)\mu(B) > 0$. From the definition of geometric mean we have $A \odot_p B \subseteq pA + (1-p)B$, for any $p \in (0, 1)$. Thus,

$$\begin{aligned} \mu(\lambda A + (1-\lambda)B) &= \mu\left(p \cdot \frac{\lambda}{p}A + (1-p) \cdot \frac{1-\lambda}{1-p}B\right) \geq \mu\left(\left(\frac{\lambda}{p}\right) \odot_p \left(\frac{1-\lambda}{1-p}B\right)\right) \\ &= \mu\left(\left(\frac{\lambda}{p}\right)^p \left(\frac{1-\lambda}{1-p}\right)^{1-p} A \odot_p B\right). \end{aligned}$$

Let $t = \left(\frac{\lambda}{p}\right)^p \left(\frac{1-\lambda}{1-p}\right)^{1-p}$ and $C = A \odot_p B$. From the concavity of the logarithm it follows that $0 \leq t \leq 1$. We have

$$\mu(tC) = \int_{tC} f(x) dx = t^n \int_C f(tx) dx \geq t^n \int_C f(x) dx = t^n \mu(C). \quad (19)$$

Therefore,

$$\mu(\lambda A + (1-\lambda)B) \geq t^n \mu(A \odot_p B) \geq t^n \mu(A)^p \mu(B)^{1-p} = \left[\left(\frac{\lambda}{p}\right)^p \left(\frac{1-\lambda}{1-p}\right)^{1-p}\right]^n \mu(A)^p \mu(B)^{1-p}.$$

Taking

$$p = \frac{\lambda \mu(A)^{1/n}}{\lambda \mu(A)^{1/n} + (1-\lambda) \mu(B)^{1/n}}$$

gives

$$\mu(\lambda A + (1-\lambda)B)^{1/n} \geq \lambda \mu(A)^{1/n} + (1-\lambda) \mu(B)^{1/n}.$$

If, say, $\mu(B) = 0$ then by (19) and the fact that $0 \in B$ we get

$$\mu(\lambda A + (1-\lambda)B)^{1/n} \geq \mu(\lambda A)^{1/n} \geq \lambda \mu(A)^{1/n} = \lambda \mu(A)^{1/n} + (1-\lambda) \mu(B)^{1/n}.$$

□

We now sketch the proof of Theorem 3.

Proof. Let $A, B \in \mathcal{K}_I$ and let us take $f, g, m : \mathbb{R}_+^n \rightarrow \mathbb{R}_+$ given by $f = \mathbf{1}_{A \cap \mathbb{R}_+^n}$, $g = \mathbf{1}_{B \cap \mathbb{R}_+^n}$ and $m = \mathbf{1}_{(A \odot_\lambda^I B) \cap \mathbb{R}_+^n}$. Let φ_μ be an unconditional log-concave density of μ . We define

$$\begin{aligned} F(x) &= f(e^{x_1}, \dots, e^{x_n}) \varphi_\mu(e^{x_1}, \dots, e^{x_n}) e^{x_1 + \dots + x_n}, & G(x) &= g(e^{x_1}, \dots, e^{x_n}) \varphi_\mu(e^{x_1}, \dots, e^{x_n}) e^{x_1 + \dots + x_n}, \\ M(x) &= m(e^{x_1}, \dots, e^{x_n}) \varphi_\mu(e^{x_1}, \dots, e^{x_n}) e^{x_1 + \dots + x_n}. \end{aligned}$$

One can easily check, using the definition of \mathcal{K}_I and the definition of the geometric mean \odot_λ^I , as well as the inequality

$$\begin{aligned} \varphi_\mu(e^{\lambda x_1 + (1-\lambda)y_1}, \dots, e^{\lambda x_n + (1-\lambda)y_n}) &= \varphi_\mu((e^{x_1})^\lambda (e^{y_1})^{1-\lambda}, \dots, (e^{x_n})^\lambda (e^{y_n})^{1-\lambda}) \\ \varphi_\mu(\lambda e^{x_1} + (1-\lambda)e^{y_1}, \dots, \lambda e^{x_n} + (1-\lambda)e^{y_n}) &\geq \varphi_\mu(e^{x_1}, \dots, e^{x_n})^\lambda \varphi_\mu(e^{y_1}, \dots, e^{y_n})^{1-\lambda}, \end{aligned}$$

that the functions F, G, M satisfy assumptions of Theorem 2. The first inequality above follows from the fact that φ_μ is unconditional. As a consequence, we get $\mu((A \odot_\lambda^I B) \cap \mathbb{R}_+^n) \geq \mu(A \cap \mathbb{R}_+^n)^\lambda \mu(B \cap \mathbb{R}_+^n)^{1-\lambda}$. The assertion follows from unconditionality of our measure μ and the fact that A, B and $A \odot_\lambda^I B$ are ideals. □

4 Examples and open problems

Example 1. The assumption, that our measure μ in Theorem 1 is product, is important. Indeed, let us take the square $C = \{|x|, |y| \leq 1\} \subset \mathbb{R}^2$ and take the measure with density $\varphi(x) = \frac{1}{2}\mathbf{1}_{2C}(x) + \frac{1}{2}\mathbf{1}_C(x)$. This density is unconditional, however non product. Let us define $\psi(a) = \sqrt{\mu(aC)}$. The assertion of Theorem 1 implies that ψ is concave. However, we have $\psi(a) = \sqrt{2a^2 + 2}$ for $a \in [1, 2]$, which is strictly convex. Thus, μ does not satisfy (6).

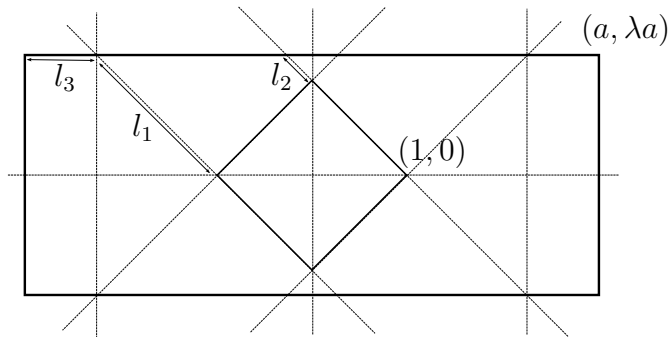
Example 2. In general, under the assumption that our measure μ is unconditional and product, one cannot prove that Theorem 1 holds true for arbitrary symmetric convex sets. To see this, let us take the product measure $\mu = \mu_0 \otimes \mu_0$ on \mathbb{R}^2 , where μ_0 has an unconditional density $\varphi(x) = p + (1-p)\mathbf{1}_{[-1/\sqrt{2}, 1/\sqrt{2}]}(x)$ for some $p \in [0, 1]$. This measure is not finite, however one can always restrict it to a cube $[-C, C]^2$ to get a finite product measure. If C is sufficiently large the example given below also produces and example for the restricted measure.

To simplify the computation let us rotate the whole picture by angle $\pi/4$. Then consider the rectangle $R = [-1, 1] \times [-\lambda, \lambda]$ for $\lambda \leq 1/2$. As in the previous example, it is enough to show that the function $\psi(a) = \sqrt{\mu(aR)}$ is not concave. Let us consider this function only on the interval $[1/\lambda, \infty)$. The condition $\lambda \leq 1/2$ ensures that the point $(a, \lambda a)$ lies in the region with density p^2 . Let us introduce lengths l_1, l_2, l_3 (see the picture below). Note that $l_1 = \sqrt{2}\lambda a$, $l_2 = \sqrt{2}(\lambda a - 1)$ and $l_3 = a - (1 + \lambda a)$. Let $\omega(a) = \mu(aR)$. We have

$$\begin{aligned} \omega(a) &= 2 + 4\sqrt{2}p \cdot \frac{l_1 + l_2}{2} + p^2 l_1^2 + p^2 l_2^2 + 4p^2 l_3 \lambda a \\ &= 2 + 4p(2\lambda a - 1) + 2p^2 \lambda^2 a^2 + 2p^2 (\lambda a - 1)^2 + 4p^2 \lambda a (a - 1 - \lambda a) \\ &= 2(1-p)^2 + 4p\lambda a (pa + 2 - 2p) = d_0 + d_1 a + d_2 a^2, \end{aligned}$$

where $d_0 = 2(1-p)^2$, $d_1 = 8p(1-p)\lambda$, $d_2 = 4p^2\lambda$. We show that ψ is strictly convex for $p \in (0, 1)$ and $0 < \lambda < 1/2$. Then $\psi'' > 0$ is equivalent to $2\omega\omega'' > (\omega')^2$. But

$$\begin{aligned} 2\omega(a)\omega''(a) - (\omega'(a))^2 &= 4d_2(d_0 + d_1 a + d_2 a^2) - (2d_2 a + d_1)^2 = 4d_2 d_0 - d_1^2 \\ &= 32\lambda p^2(1-p)^2 - 64\lambda^2 p^2(1-p)^2 = 32\lambda p^2(1-p)^2(1-2\lambda) > 0. \end{aligned}$$



Let us state some open questions that arose during our study.

Question. Let us assume that the measure μ has an even log-concave density (not-necessarily product). Does the assertion of Theorem 1 holds true for arbitrary symmetric sets A and B ? If not, is it true under additional assumption that the sets are unconditional or that the measure is product? In particular, can one remove the assumption of unconditionality in the Gaussian Brunn-Minkowski inequality?

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