A WEAKLY CHAINABLE UNIQUELY ARCWISE CONNECTED CONTINUUM WITHOUT THE FIXED POINT PROPERTY

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G.S.Young in [Yo] constructed a uniquely arcwise connected continuum without the fixed point property. D.P.Bellamy asked in [Be] whether such a continuum can be weakly chainable, that means a continuous image of a chainable continuum. Here we sketch an example.

Let us consider in the three-dimensional space $\mathbb{R}^3$ a subset $A = \{(x, 0, z) \in \mathbb{R}^3 : z = \sin \frac{\pi}{2x}, x \in (0, 1]\} \cup I$, where $I = \{(0, 0, z) : z \in [-1, 1]\}$ is the density segment of the $\sin \frac{1}{x}$-curve A, let B be the image of A under the symmetry in the plane $x = 1$ and let $C = A \cup B$, a symmetric $\sin \frac{1}{x}$-curve. Denote $D = C \cup (2, 0, 0) + C$, where $v + X$ means set X shifted by vector v. Let $D_n = (0, \frac{1}{n}, \frac{1}{n}) + D$ for $n = 1, 2, ...$ and $\mathfrak{D} = \bigcup_{n=1}^{\infty} D_n \cup D$. We can define on $\mathfrak{D}$ a decomposition with the following layers :the sets of the form $\{d_1, d_2\}$, where $d_1 \in D_{2k-1}$, $d_2 \in D_{2k}$, the first coordinates of d_1 and d_2 are equal to 0 and their third coordinates are equal each to other,for $k = 1, 2, \ldots$; the sets of the form $\{d_1, d_2\}$, where $d_1 \in D_{2k+1}, d_2 \in D_{2k}$ are points with first coordinates equal to 4, for and their third coordinates equal each to other, for $k = 1, 2 \ldots$; remaining layers are one point sets. The decomposition space is a chainable continuum $\mathfrak{S}$. Consider the orthogonal projection p of $\mathfrak{D}$ onto the plane $y = 0$. It agrees with the decomposition on $\mathfrak{D}$ hence $\mathcal{D} = p(\mathfrak{D})$ is a continuous image of $\mathfrak{S}$ (it is a double ladder which rungs are symmetric $\sin \frac{1}{x}$-curves). Now, let h be a homeomorphism which maps the segment $\{(4, 0, z) : z \in [-1, 2]\}$ onto the segment $\{(0, 0, z) : z \in [-1, 1.5]\}$, such that $h(4, 0, 1 + \frac{1}{n}) = (0, 0, 1 + \frac{1}{n+1})$ and $h(4, 0, -1 + \frac{1}{n}) = (0, 0, -1 + \frac{1}{n+1})$ for $n = 1, 2, \ldots$.Consider a decomposition space obtained from $\mathcal{D}$ by gluing together some points. Namely, we glue the points $(4, 0, z)$ and $h(4, 0, z)$ together for $z \in [-1, 2]$. Moreover each point of the form $(2 - \frac{1}{4n}, 0, \frac{1}{n})$ is glued together with the point $(2 + \frac{1}{4(n+1)}, 0, \frac{1}{n})$ and each point of the form $(4 - \frac{1}{4(n+1)}, 0, \frac{1}{n})$ is glued together with the point $(\frac{1}{4(n+2)}, 0, \frac{1}{n+1})$ for $n = 1, 2, 3 \ldots$-we will call them points of connection. Denote $\mathcal{D}'$ the decomposition space. Our example is $X = \mathcal{D}' \cup P$, where P is a pentod joining its branch point w with $(1, 0, 2)$, $(0, 0, -1)$(equal to $(4, 0, -1)$),$(2, 0, -1)$, $(1, 0, 1)$, and $(3, 0, 1)$ in $\mathcal{D}'$ and elsewhere disjoint with $\mathcal{D}'$. Let us remark that X is weakly chainable-it is a continuous image of $\mathcal{D}'$ since we can mould the pentod P from an arc containing the point $(1, 0, 2)$. It is easy to check that obtained space is uniquely arcwise connected. The fixed point free mapping f is a modification of the Young's one. On the double warsaw circle S obtained from D it is the composition of a rotation which maps C and $(2, 0, 0) + C$ mutually each onto other with a mapping which stretches properly a central arc of each of the symmetric $\sin \frac{1}{x}$-curves in S and shifts outwards remaining folds through a one. The branches of the pentod joining w with the points

$(0,0,-1),(2,0,-1),(3,0,1),(1,0,1)$ are permuted in accordance with rotation of S and then stretched such that the branch point w goes onto $(1,0,2)$ pushing the fifth branch onto the arc joining $(1,0,2)$ with the point $(3,0,2)$. The mapping extends to the rest of the space as follows: it pushes each of the remaining symmetric $\sin\frac{1}{x}$-curves to the adjacent(i.e.,having a common density segment) next one (starting from $(0,0,1)+B$) by a homeomorphism preserving points of connection, adjusted to the action on S.

References

[Be] D.P.Bellamy, *Fixed point in dimension one*, Continua with the Houston Problem Book, Dekker New York, 1995.

[Yo] G.S.Young, *Fixed point theorems for arcwise connected continua*, Proc.Amer.Math.Soc. **11** (1960), 880-884.

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