ON PROBABILISTIC QUANTIFIED SATISFIABILITY GAMES

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Abstract. We study the complexity of some new probabilistic variant of the problem Quantified Satisfiability (QSAT). Let a sentence \( \exists v_1 \forall v_2 \ldots \exists v_{n-1} \forall v_n \phi \) be given. In classical game associated with the QSAT problem, the players \( \exists \) and \( \forall \) alternately chose Boolean values of the variables \( v_1, \ldots, v_n \). In our game one (or both) players can instead determine the probability that \( v_i \) is true. We call such player a probabilistic player as opposite to classical player. The payoff (of \( \exists \)) is the probability that the formula \( \phi \) is true. We study the complexity of the problem if \( \exists \) (probabilistic or classical) has a strategy to achieve the payoff at least \( c \) playing against \( \forall \) (probabilistic or classical). We completely answer the question for the case of threshold \( c = 1 \), exhibiting that the case when \( \forall \) is probabilistic is easier to decide (\( \Sigma^P_2 \)-complete) than the remaining cases (PSPACE-complete).

For thresholds \( c < 1 \) we have a number of partial results. We establish PSPACE-hardness of the question whether \( \exists \) can win in the case when only one of the players is probabilistic, and \( \Sigma^P_2 \)-hardness when both players are probabilistic. We also show that the set of thresholds \( c \) for which a related problem is PSPACE is dense in \([0,1]\).

We study the set of reals \( c \in [0,1] \) that can be game values of our games. The set turns out to include the set of binary rationals, but also some irrational numbers.

1. Introduction

In this paper we study a certain probabilistic variant of the problem Quantified Satisfiability (QSAT). Games with coin tosses (see e.g. [11][5][4][7]) or the games where players use randomized strategies (see e.g. [8][13][6]), have been widely considered in several previous works in complexity theory. Many papers consider a possibility of players to choose probability distributions (mixed strategies [13][6][9][10] or behavior strategies[8][9]), but the choices are made by the players just once per game, either independently or with just one alternation. A crucial difference between these works and ours is that in our framework probabilities are chosen by players in turn, according to the values of probabilities chosen so far. To our knowledge, such situation has not been considered so far.

Quantified Satisfiability was studied in [2]. It can be considered as a game between two players, call them \( \exists \) and \( \forall \). Fix some Boolean formula \( \phi(x_1, \ldots, x_n) \). The two players move alternately, with \( \exists \) moving first. If \( i \) is odd then \( \exists \) fixes the value of \( x_i \), whereas if \( i \) is even \( \forall \) fixes the value of \( x_i \). \( \exists \) tries to make the expression \( \phi \) true, while \( \forall \) tries to make it false. Then \( \exists \) has a winning strategy iff \( \exists x_1 \forall x_2 \exists x_3 \ldots \phi(x_1, \ldots, x_n) \) is true. If we assume that \( \forall \) is uninterested in winning and plays at random then the game becomes a Game Against Nature studied in [11](see also [12]). The decisions of the Nature are probabilistic in manner, Nature chooses \( x_i = 0 \) or \( x_i = 1 \) with probability \( \frac{1}{2} \). In this case a winning strategy for \( \exists \) is a strategy that enforces the probability of success to be greater than \( \frac{1}{2} \).
Both in the case of the game Quantified Satisfiability and in the case of the Game Against Nature the following problem is PSPACE-complete[11]: Given $\phi$ decide whether there exists a winning strategy for $\exists$.

There will be a difference between the Games Against Nature and our probabilistic variant of the game Quantified Satisfiability. In Games Against Nature the players use deterministic (pure) strategies. It means that at a particular node in a game, a player $\exists$ (playing against Nature) is required to make a strategic move - say to choose the side of the coin. Or else, Nature is required to toss a coin, but the probabilities associated with the coin tosses are fixed in advance and not chosen by Nature. Hence, coin tosses correspond to "chance moves" in standard game-theoretic terminology.

In our game, the biases of the coins will be chosen strategically in turn by both players. Once the biases of all the coins are determined, the coins are tossed. Thus the values of $x_1, \ldots, x_n$ are determined and $\exists$ wins iff $\phi(x_1, \ldots, x_n)$ is true.

More specifically, we consider two types of players. A probabilistic player, instead of determining the value of $x_i$, chooses the probability $p_i$ of $x_i$ being 1, where we assume that events $\{x_i = 1\}$ and $\{x_j = 1\}$ are independent when $i \neq j$. A player using a classical strategy, i.e. choosing values 0 or 1, can be viewed as a probabilistic player as well, but restricted to $p_i = 0$ or $p_i = 1$. The chosen probabilities $p_1, p_2, \ldots, p_n$ determine the probability $P(\phi)$ of the event that $\phi$ is true. Now $\exists$ tries to make $P(\phi)$ as large as possible whereas $\forall$ tries to minimize $P(\phi)$. So $P(\phi)$ can be meant as the payoff of $\exists$. Notice that in the classical Quantified Satisfiability game the payoff $P(\phi)$ can be only 0 or 1. The following computational problem arises: given formula $\phi$ decide if $\exists$ can make $P(\phi)$ greater than a fixed threshold $c \in [0, 1)$.

We shall study the problem and related ones in this paper. We prove that the problem if $\exists$ can make $P(\phi) = 1$ is $\Sigma_2^P$-complete (see e.g. [12]), when $\forall$ is probabilistic, and that this question is PSPACE-complete, when $\forall$ is classical. We show that it is PSPACE-hard to tell whether a probabilistic $\exists$ can enforce $P(\phi) \geq c$, when the opponent $\forall$ is classical. Similarly it is PSPACE-hard to tell whether a classical $\exists$ can make $P(\phi) > c$ when $\forall$ is probabilistic. In both cases we assume that thresholds are fixed. We also present Poly$(|\phi|, |\log_2 \varepsilon|)$-space algorithm which, given $\phi$ and $\varepsilon > 0$, returns a value that is $\varepsilon$-close to the maximal value of $P(\phi)$ attainable by $\exists$. We prove that for all $c \in \{>, \geq\}$ and for all types of players $\exists$ and $\forall$ (classical or probabilistic) the following set is dense in $[0, 1]$: the set of constants $c \in [0, 1]$ such that the language of Boolean formulas $\phi$ such that $\exists$ can make $P(\phi) < c$ is in PSPACE.

2. Variants Of The Problem Of Quantified Satisfiability

Let $V$ be a countable set of variables. Recall a definition of the set of Boolean formulas

$$\Phi ::= 0 \mid 1 \mid V \mid \sim \Phi \mid (\Phi \lor \Phi) \mid (\Phi \land \Phi).$$

Fix $\phi(v_1, \ldots, v_n) \in \Phi$. Let $x_i \in \{0, 1\}, 1 \leq i \leq n$. Then the meaning of $\phi(x_1, \ldots, x_n)$ is the logical value of $\phi$ after replacing variables $v_1, \ldots, v_n$ in $\phi$ by $x_1, \ldots, x_n$ respectively. Now let $X_1, \ldots, X_n$ be pairwise independent random variables with range $\{0, 1\}$. Naturally $\phi(X_1, \ldots, X_n)$ can be understood as the random variable with range $\{0, 1\}$ such that $P(\phi(X_1, \ldots, X_n) = 1)$, also written $P(\phi(X_1, \ldots, X_n))$ for short, equals the probability of the event that $(X_1, \ldots, X_n)$
satisfies $\phi$:

\begin{equation}
(1) \quad P(\phi(X_1, \ldots, X_n) = 1) = \sum_{(x_1, \ldots, x_n) \in \{0,1\}^n} \prod_{i=1}^n P(X_i = x_i).
\end{equation}

Note that $P(\phi(X_1, \ldots, X_n) = 1)$ is the expected value of $\phi(X_1, \ldots, X_n)$. In the sequel, $P_{p_1, \ldots, p_n}(\phi)$ stands for $P(\phi(X_1, \ldots, X_n) = 1)$, where $X_i$s are arbitrary pairwise independent random variables satisfying $P(X_i = 1) = p_i$, $1 \leq i \leq n$. For all $p_1, \ldots, p_n \in [0,1]$

\begin{equation}
(2) \quad P_{p_1, \ldots, p_n}(\phi) = \sum_{(x_1, \ldots, x_n) \in \{0,1\}^n} \prod_{i=1}^n p_i(x_i)
\end{equation}

where $p_i(x_i) = \begin{cases} p_i & \text{if } x_i = 1 \\ 1 - p_i & \text{if } x_i = 0 \end{cases}$.

For the rest of this paper we shall assume that the range of random variables we consider is $\{0,1\}$ and that differently named random variables are pairwise independent. For instance $X_1$ and $X_2$ would denote two pairwise independent random variables with range $\{0,1\}$. We shall write $\phi(X_1, \ldots, X_n)$ as the abbreviation for $P(\phi(X_1, \ldots, X_n) = 1) = 1$.

Consider the following statement: "There is a random variable $X$ such that for every random variable $Y$ we have $P(X \leftrightarrow Y) \geq \frac{1}{2}$" (Here we wrote $\phi_1 \leftrightarrow \phi_2$ as the abbreviation for $((\phi_1 \lor \phi_2) \land (\sim \phi_1 \lor \phi_2)))$. It is a true statement - consider random variable $X$ with $P(X = 1) = \frac{1}{2}$. This statement can be rewritten as

\[ \exists X \forall Y \ P(X \leftrightarrow Y) \geq \frac{1}{2}. \]

We used uppercase letters $X$ and $Y$ to emphasize that they represent random variables. Sometimes we would like to state also that: "There is a random variable $X$ such that for every $y \in \{0,1\}$ we have $P(X \leftrightarrow y) \geq \frac{1}{2}$." This can be viewed as the previous statement with $Y$ restricted to two random variables: such that $P(Y = 1) = 1$ or $P(Y = 0) = 1$. We will denote it by

\[ \exists X \forall y \ P(X \leftrightarrow y) \geq \frac{1}{2}. \]

Here and subsequently, $\exists X$ means that there is a random variable $X$, $\exists x$ means that there is a random variable $x$ restricted to two random variables: such that $P(x = 1) = 1$ or $P(x = 0) = 1$. Similarly in the case of quantifier $\forall$. We extend this notation to longer prefixes in obvious way. Note that $\exists x_1 \forall y_1 \exists x_2 \ldots \phi$ has its usual meaning.

Consider formula of the form:

\begin{equation}
(3) \quad Q_{1y_1} Q_{2y_2} Q_{3y_3} \ldots Q_{ny_n} P(\phi(y_1, y_2, y_3, \ldots, y_n)) < c
\end{equation}

where $< \in \{\geq, \leq, >, <\}$, $c \in [0,1]$, $y_i \in \{x_i, X_i\}$, $Q_i \in \{\exists, \forall\}$, $1 \leq i \leq n$. We will interpret formula (3) as the game between $\exists$ and $\forall$. A player $Q_i$ chooses $y_i$ in turn, for $i = 1, \ldots, n$. $\exists$ wins if $P(\phi(y_1, y_2, y_3, \ldots, y_n)) < c$, after all $y_is$ are chosen. $\exists$ has a winning strategy, if he can make $P(\phi(y_1, y_2, y_3, \ldots, y_n)) < c$, and $\forall$ has a winning strategy, if he can make $P(\phi(y_1, y_2, y_3, \ldots, y_n)) \geq c$. Obviously $\exists$ has a winning strategy iff formula (3) is true, and $\forall$ has a winning strategy iff the following formula is true:

\begin{equation}
Q_{1y_1} Q_{2y_2} Q_{3y_3} \ldots Q_{ny_n} P(\phi(y_1, y_2, y_3, \ldots, y_n)) \geq c
\end{equation}

where $Q_i$ is $\exists$ if $Q_i = \forall$, and $Q_i$ is $\forall$ if $Q_i = \exists$, $1 \leq i \leq n$. In the case of being $'\geq'$ or $'>'$ $\exists$ tries to make $P(\phi(y_1, y_2, y_3, \ldots, y_n))$ as big as possible, and then
it is natural to call $P (\phi (y_1, y_2, y_3, \ldots, y_n))$ the payoff of $\exists$. If $y_i = X_i$ for every $y_i$ chosen by $\exists$, then we call $\exists$ a probabilistic player, and we say that he uses a probabilistic strategy. If $y_i = x_i$ for every $y_i$ chosen by $\exists$, then we call $\exists$ a classical player, and we say he uses a classical strategy then. We use similar terminology for the case of the player $\forall$.

For the rules of the game described in the introduction we can consider following problem.

**Problem 1.** Fix $c \in [0, 1)$. Given Boolean formula $\phi$ decide whether

$$\exists X_1 \forall X_2 \exists X_3 \ldots \exists_n X_n \ P (\phi (X_1, X_2, X_3, \ldots, X_n)) > c$$

where the $n$th quantifier $\exists_n$ is $\exists$ if $n$ is odd, and $\forall$ if $n$ is even.

In the case of threshold $c$ given by finitely representable rational number decidability of the problem 1 and of similar ones follows from Tarski’s Theorem on the decidability of the first-order theory of the field of real numbers. For example, we can rewrite formula $\exists X \forall Y \ P (X \leftrightarrow Y) \geq \frac{1}{2}$ as the following sentence of theory of reals

$$\exists p_X (0 \leq p_X \leq 1) \land \forall p_Y [(0 \leq p_Y \leq 1) \Rightarrow p_X p_Y + (1 - p_X)(1 - p_Y) > c] .$$

In general, the size of an expression representing $P (\phi (X_1, X_2, X_3, \ldots, X_n))$ can be of exponential size with respect to the size of $\phi$.

The following problem is PSPACE-complete[2].

**Problem 2** (Quantified Satisfiability). Given formula $\phi$ decide whether

$$\exists x_1 \forall x_2 \exists x_3 \ldots \exists_n x_n \phi .$$

One may make conjecture that $\exists X_1 \forall X_2 \exists X_3 \ldots \exists_n X_n \phi (X_1, X_2, X_3, \ldots, X_n)$ is equivalent to $\exists x_1 \forall x_2 \ldots \exists_n x_n \phi (x_1, x_2, x_3, \ldots, x_n)$. But it is not true as the following example shows.

**Example 1.** Let $\phi = v_1 \leftrightarrow v_2$. Then $\forall x_1 \exists x_2 \phi (x_1, x_2)$ is true but it is not true that $\forall x_1 \exists x_2 \phi (X_1, X_2)$, because if $P (X_1 = 1) = \frac{1}{2}$, then

$$P (\phi (X_1, X_2)) = P (X_1 = 0) P (X_2 = 0) + P (X_1 = 1) P (X_2 = 1) = \frac{1}{2} P (X_2 = 0) + \frac{1}{2} P (X_2 = 1) = \frac{1}{2} < 1$$

whenever $X_2$ is chosen. \hfill $\Box$

The next example shows that for some Boolean formulas $\phi$ quantified formula $\exists x_1 \forall x_2 \ldots \exists_n x_n \phi$ is true whereas $\exists X_1 \forall X_2 \ldots \exists_n X_n \ P (\phi (X_1, \ldots, X_n)) \geq c$ is true only when $c$ is negligible.

**Example 2.** Let $\phi = \bigwedge_{i=1}^{n} (v_{2i-1} \leftrightarrow v_{2i})$. Then $\forall x_1 \exists x_2 \ldots \exists_{2n-1} \exists_{2n} \phi (x_1, \ldots, x_{2n})$ is true but $\forall X_1 \exists X_2 \ldots \exists_{2n-1} \exists_{2n} \ P (\phi (X_1, \ldots, X_{2n})) \geq c$ is not true unless $c \leq \frac{1}{2^n}$. To see this observe that if $P (X_{2i-1} = 1) = \frac{1}{2}$ for all $1 \leq i \leq n$, then $P (X_{2i-1} \leftrightarrow X_{2i}) = \frac{1}{2}$ for all $1 \leq i \leq n$ (see the previous example) and in consequence $P (\phi (X_1, \ldots, X_{2n})) = \prod_{i=1}^{n} P (X_{2i-1} \leftrightarrow X_{2i}) = \frac{1}{2^n}$, no matter how $\forall$ chooses $X_2, \ldots, X_{2n}$. We used the fact that for arbitrary Boolean formulas $\phi_1 (v_1, \ldots, v_n)$ and $\phi_2 (w_1, \ldots, w_m)$

$$P (\phi_1 (X_1, \ldots, X_n) \land \phi_2 (Y_1, \ldots, Y_m)) = P (\phi_1 (X_1, \ldots, X_n)) P (\phi_2 (Y_1, \ldots, Y_m))$$

when $X_i$s and $Y_i$s are pairwise independent random variables. \hfill $\Box$
The example above may seem to suggest that if a player has no winning strategy then the best he can do is to always choose probability $\frac{1}{2}$. But the following example illustrates that this need not be the case.

**Example 3.** Consider formula $\phi(v_1, v_2, v_3, v_4)$ such that $\phi(x_1, x_2, x_3, x_4)$ is true if and only if

$$ (x_1, x_2, x_3, x_4) \in \{(1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (0, 1, 1, 0), (1, 1, 0, 0), (1, 0, 0, 1), (0, 1, 0, 1), (1, 1, 0, 1)\}. $$

One can check that $\exists x_1 \forall x_2 \exists x_3 \forall x_4 \phi(x_1, x_2, x_3, x_4)$ is not true. The value $F(p)$ defined by

$$ F(p) = \begin{cases} p & \text{if } 0 \leq p \leq \frac{1}{2} \\ \frac{1}{2} - \frac{3p + 6p^2 + \sqrt{-1 - 2p + p^2 + 1}}{2} & \text{if } \frac{1}{2} < p \leq 1 \end{cases} $$

represents the maximal value of $P(\phi)$ available to $\exists$, when $\exists$ chooses $P(X_1 = 1) = p$ in the first move. The computation is explained in Example 4 below. The value of $p$ maximizing $F(p)$ is $p^* = \frac{1}{6} \sqrt{(53 - 6\sqrt{78}) + \frac{1}{6} \sqrt{(53 + 6\sqrt{78}) - \frac{1}{6} \sqrt{78}}} \approx 0.657298 \neq \frac{1}{2}$.

If $\exists$ would choose $P(X_1 = 1) = \frac{1}{2}$ instead, then he could attain $F(\frac{1}{2}) = \frac{1}{2}$ at most, which is less than $F(p^*) \approx 0.553906$. It is worth noting that both $p^*$ and $F(p^*)$ are irrational\(^1\).

It may come as a slight surprise, that the problem to decide whether $\exists X_1 \forall X_2 \exists X_3 \ldots \exists_n X_n \phi$ is not as hard as QSAT, unless PSPACE collapses to the second level of the polynomial hierarchy (see also Summary at page 7):

**Theorem 1.**

$$ \exists x_1 \exists x_3 \ldots \exists x_{n/2} \forall x_4 \ldots \forall x_n \phi $$

$\iff$

$$ \exists x_1 \forall X_2 \exists x_3 \ldots \exists_n \chi_n \phi $$

$\iff$

$$ \exists x_1 \forall X_2 \exists x_3 \ldots \exists_n \chi_n P(\phi) > 1 - \frac{1}{2^{n/2}} $$

$\iff$

$$ \exists X_1 \forall X_2 \exists X_3 \ldots \exists_n X_n \phi $$

$\iff$

$$ \exists X_1 \forall X_2 \exists X_3 \ldots \exists_n X_n P(\phi) > 1 - \frac{1}{2^{n/2}} $$

where $\chi_n$ is $x_n$ if $n$ is odd, and $X_n$ if $n$ is even, and $\iota = n, \kappa = n - 1$ if $n$ is odd, and $\iota = n - 1, \kappa = n$ if $n$ is even.

**Proof.** See Appendix A \(\square\)

The following theorem shows that if the player $\forall$ is classical then probabilistic strategy does not add the power to $\exists$, when threshold $c$ is set to 1.

**Theorem 2.**

$$ \exists x_1 \forall x_2 \exists x_3 \ldots \exists_n x_n \phi $$

$\iff$

$$ \exists X_1 \forall x_2 \exists X_3 \ldots \exists_n \chi_n \phi $$

$\iff$

$$ \exists X_1 \forall x_2 \exists X_3 \ldots \exists_n \chi_n P(\phi) > 1 - \frac{1}{2^{n/2}} $$

where $\chi_n$ is $X_n$ if $n$ is odd, and $x_n$ if $n$ is even.

**Proof.** See Appendix B. \(\square\)

\(^1\)We used command `FullSimplify[x ∈ Rationals]` in Mathematica ver. 4.0.1.0 program created by Wolfram Research, Inc., to get this result.
3. Game Value

**Definition.** Let \( \phi (v_1, \ldots, v_n) \in \Phi \).

\[
(5) \quad c_\phi = \max_{p_1 \in [0,1]} \min_{p_2 \in [0,1]} \ldots \min_{p_n \in [0,1]} P_{p_1,\ldots,p_n} (\phi)
\]

\[
(6) \quad c_\phi' = \max_{p_1 \in [0,1]} \min_{p_2 \in [0,1]} \ldots \min_{p_n \in \Delta_n} P_{p_1,\ldots,p_n} (\phi)
\]

\[
(7) \quad c_\phi'' = \max_{p_1 \in [0,1]} \min_{p_2 \in [0,1]} \ldots \min_{p_n \in \Lambda_n} P_{p_1,\ldots,p_n} (\phi)
\]

where \( \Box_n, \Delta_n, \Lambda_n \) are max, \( [0,1] \), \( \{0,1\} \) respectively if \( n \) is odd, and min, \( [0,1] \), \( [0,1] \) if \( n \) is even.

Let

\[
\Gamma = \{ c_\phi : \phi \in \Phi \}, \Gamma' = \{ c_\phi' : \phi \in \Phi \}, \Gamma'' = \{ c_\phi'' : \phi \in \Phi \}.
\]

The values at the right-hand sides of the formulas (5), (6) and (7), call them the game values, are well defined because the sets \( [0,1] \) and \( \{0,1\} \) are compact and for \( 1 < i \leq n \) the following maps are continuous with respect to \( p_1, \ldots, p_i \):

\[
p_1, \ldots, p_i \mapsto \Box_{i+1} \ldots \Box_n P_{p_1,\ldots,p_n} (\phi)
\]

\[
p_1, \ldots, p_i \mapsto \Box_{i+1} \ldots \Box_n P_{p_1,\ldots,p_n} (\phi)
\]

\[
p_1, \ldots, p_i \mapsto \Box_{i+1} \ldots \Box_n P_{p_1,\ldots,p_n} (\phi)
\]

\( P_{p_1,\ldots,p_n} (\phi) \) is continuous (case \( i = n \)) because it is a multilinear map (recall (2)). The continuity of maps in case when \( i < n \) can be inductively proved by the use of the following lemma.

**Lemma 1.** Assume \( f : S \times T \to \mathbb{R} \) is a continuous map and \( S, T \) are compact spaces. Then \( F \) defined by \( F(s) = \max_{t \in T} f(s,t) \) is also continuous.

**Proof.** See Appendix C.

The values \( c_\phi, c_\phi', c_\phi'' \) defined by (5), (6) and (7) are the maximal attainable payoffs of \( \exists \) in corresponding games. To see this observe that if \( f(p) \) is the payoff of the player corresponding to a choice \( p \in P \), where \( P \) is the compact set of all possible choices, then \( F = \max_{p \in P} f(p) \) is the maximal attainable payoff of the player provided \( f \) is a continuous map.

**Example 4.** Let \( \phi \) be as in example 3. Let

\[
A (p_1, p_2, p_3) = -2p_1 p_2 p_3 - p_3 + p_1 p_3 + p_1 p_2
\]

\[
B (p_1, p_2, p_3) = p_3 - p_2 p_3 - 2p_1 p_3 + 3p_1 p_2 p_3 + p_1 - 2p_1 p_2 + p_2
\]

\[
A_1 (p_1, p_2) = 3p_1 p_2 + 1 - p_2 - 2p_1
\]

\[
B_1 (p_1, p_2) = p_1 - 2p_1 p_2 + p_2
\]

\[
A_2 (p_1, p_2) = -p_2 + p_1 p_2 - p_1
\]

\[
B_2 (p_1, p_2) = p_1 + p_2 - p_1 p_2.
\]
We compute $c_\phi$.

\[
P_{p_1,p_2,p_3,p_4}(\phi) = A(p_1,p_2,p_3) p_4 + B(p_1,p_2,p_3)
\]

\[
\min_{p_4 \in [0,1]} P_{p_1,p_2,p_3,p_4}(\phi) = \begin{cases} 
A_1(p_1,p_2) p_3 + B_1(p_1,p_2) & \text{if } A(p_1,p_2,p_3) \geq 0 \\
A_2(p_1,p_2) p_3 + B_2(p_1,p_2) & \text{if } A(p_1,p_2,p_3) < 0
\end{cases}
\]

\[
\max \min_{p_3 \in [0,1]} P_{p_1,p_2,p_3,p_4}(\phi) = \begin{cases} 
A_1(p_1,p_2) \frac{p_4}{1-p_1+p_4} + B_1(p_1,p_2) & \text{if } A_1(p_1,p_2) \geq 0 \\
B_1(p_1,p_2) & \text{if } A_1(p_1,p_2) < 0
\end{cases}
\]

\[
\min \max \min_{p_2 \in [0,1]} P_{p_1,p_2,p_3,p_4}(\phi) = F(p_1)
\]

\[
\max \max \min_{p_1 \in [0,1]} P_{p_1,p_2,p_3,p_4}(\phi) = F(p^*) \approx 0.553906
\]

where $F$ and $p^*$ are defined in example 3.

\[\square\]

**Example 5.** Let $\phi$, $A$, $B$, $A_1$, $B_1$, $A_2$, $B_2$ be as in example 4. We compute $c'_\phi$.

\[
P_{p_1,p_2,p_3,p_4}(\phi) = A(p_1,p_2,p_3) p_4 + B(p_1,p_2,p_3)
\]

\[
\min_{p_4 \in [0,1]} P_{p_1,p_2,p_3,p_4}(\phi) = \begin{cases} 
A_1(p_1,p_2) p_3 + B_1(p_1,p_2) & \text{if } A(p_1,p_2,p_3) \geq 0 \\
A_2(p_1,p_2) p_3 + B_2(p_1,p_2) & \text{if } A(p_1,p_2,p_3) < 0
\end{cases}
\]

\[
\max \min_{p_3 \in [0,1]} P_{p_1,p_2,p_3,p_4}(\phi) = \begin{cases} 
A_1(p_1,p_2) \frac{p_4}{1-p_1+p_4} + B_1(p_1,p_2) & \text{if } A_1(p_1,p_2) \geq 0 \\
B_1(p_1,p_2) & \text{if } A_1(p_1,p_2) < 0
\end{cases}
\]

\[
\min \max \min_{p_2 \in [0,1]} P_{p_1,p_2,p_3,p_4}(\phi) = \begin{cases} 
p_1 \frac{p_4}{1+p_1} & \text{if } p_1 \leq p^*_1 \\
p_1 & \text{if } p_1 > p^*_1
\end{cases}
\]

\[
\max \max \min_{p_1 \in [0,1]} P_{p_1,p_2,p_3,p_4}(\phi) = p_1^* \approx 0.618034
\]

where $p_1^* = \frac{1}{2} (\sqrt{5} - 1)$.

\[\square\]

One can easily check that for every formula $\phi$ the following equations hold, relating the game values for $\phi$ and $\sim \phi$.

\[
1 = c_{\phi(v_1,\ldots,v_n)} + c_{\sim \phi(v_0,v_1,\ldots,v_n)}
\]

\[
(8) = c'_{\phi(v_1,\ldots,v_n)} + c''_{\phi(v_0,v_1,\ldots,v_n)} = c'_{\phi(v_0,v_1,\ldots,v_n)} + c'_{\phi(v_1,\ldots,v_n)}
\]

where we used dummy variable $v_0$ not used in formula $\phi$ to enforce $x_1$ or $X_1$ (according to the type of the game), be chosen by $\forall$. Observe that by (8) we have $\Gamma'' = \{1 - \gamma : \gamma \in \Gamma'\}$.

We also have the following inequalities.

\[
c''_{\phi(v_1,\ldots,v_n)} \leq c_{\phi(v_1,\ldots,v_n)} \leq c'_{\phi(v_1,\ldots,v_n)}
\]

**Theorem 3.** For every $c \in \Gamma' \setminus \{0\}$ the following problem is PSPACE-hard:

Given $\phi$ decide whether $\exists X_1 \forall x_2 \exists X_3 \ldots \exists \chi_n P(\phi) \geq c$.

**Proof.** See Appendix D.

\[\square\]

**Theorem 4.** For every $c \in \Gamma'' \setminus \{1\}$ the following problem is PSPACE-hard:

Given $\phi$ decide whether $\exists x_1 \forall X_2 \exists x_3 \ldots \exists \chi_n P(\phi) > c$.

**Proof.** See Appendix E.

\[\square\]

Theorems 1, 2, 3, 4 are summarized below. We will rephrase them in game-theoretic terms. That is, the problem concerning $\exists X_1 \forall x_2 \exists X_3 \ldots \exists \chi_n P(\phi) > c$ is considered as the problem of $\exists$ using probabilistic strategy, against $\forall$ using classical strategy. Similarly for other cases.

**Summary of the complexity results.** Assume $\phi$ is given and $c$ is arbitrary fixed number $c \in [0,1]$, until otherwise stated. We put three questions: if $\exists$ can make: (i) $P(\phi) = 1$, (ii) $P(\phi) > c$, (iii) $P(\phi) \geq c$. Our complexity results when one or
both players are probabilistic) depend on the natures of strategies that both players use. (Of course if both players are classical the results are obvious consequences of the PSPACE-completeness of QSAT.)

\[
P(\phi) = 1
\]

<table>
<thead>
<tr>
<th>Probabilistic</th>
<th>Classical</th>
<th>Probabilistic</th>
<th>Classical</th>
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</thead>
<tbody>
<tr>
<td>PSP\text{-}complete</td>
<td>Σ_2^P-complete</td>
<td>PSP\text{-}complete</td>
<td>Σ_2^P-complete</td>
</tr>
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</table>

\[
P(\phi) > c
\]

<table>
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<th>Classical</th>
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<tbody>
<tr>
<td>PSP\text{-}hard*</td>
<td>Σ_2^P-hard</td>
<td>PSP\text{-}hard**</td>
<td>Σ_2^P-hard***</td>
</tr>
</tbody>
</table>

\[
P(\phi) \geq c
\]

<table>
<thead>
<tr>
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<th>Classical</th>
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</thead>
<tbody>
<tr>
<td>Σ_2^P-hard***</td>
<td>?</td>
<td>Σ_2^P-hard***</td>
<td>?</td>
</tr>
</tbody>
</table>

* when \( c \) is the part of an input
** when \( c \in \Gamma'' \setminus \{1\} \)
*** when \( c \in \Gamma' \setminus \{0\} \)

The next theorem yields a partial information concerning the shape of the sets \( \Gamma, \Gamma' \) and \( \Gamma'' \). A number \( b \) is binary rational if \( b = \sum_{i=1}^{n} b_i \frac{1}{2^i} \) for some \( n \) and for some \( b_1, \ldots, b_n \in \{0, 1\} \). Let \( \Upsilon \) be the set of all binary rationals in \([0, 1]\).

**Theorem 5.** \( \Upsilon \subseteq \Gamma \), \( \Upsilon \subseteq \Gamma' \) and \( \Upsilon \subseteq \Gamma'' \).

**Proof.** See Appendix F. \(\square\)

**Corollary.** The sets \( \Gamma, \Gamma' \) and \( \Gamma'' \) are dense subsets of the interval \([0, 1]\).

We say that \( \lambda' \) is \( \varepsilon \)-close to \( \lambda \) if \( |\lambda - \lambda'| \leq \varepsilon \).

**Theorem 6.** Let \( \Delta_i = [0, 1] \) or \( \Delta_i = \{0, 1\} \) for every \( 1 \leq i \leq n \). Given \( \phi(x_1, \ldots, x_n) \) and \( \varepsilon > 0 \), we can compute in \( O(\log_2 |\phi| + n \log_2 n + n |\log_2 \varepsilon|) \) space a number \( \lambda' \) that is \( \varepsilon \)-close to \( \lambda = \max \min \max \ldots \min \prod_{p_1 \in \Delta_1, p_2 \in \Delta_2, \ldots, p_n \in \Delta_n} P_{p_1,\ldots,p_n}(\phi) \). In particular, we can compute the approximation of game values \( c_\phi, c'_\phi, c''_\phi \) within the bound just mentioned.

**Proof.** See Appendix G. \(\square\)

One may ask if Theorem 6 could be used to solve Problem 1 in polynomial space, at least for some \( c \). Lemma 2 enables us to give an affirmative answer to this question.

**Lemma 2.** Let \( D \subseteq \Sigma^* \) be a language over a finite alphabet \( \Sigma \), \( |\Sigma| \geq 2 \), and let \( P \) be a map \( P : D \to [0, 1] \). Assume for given \( d \in D \) we can compute in space \( O(\text{Poly}(|d|, |\log \varepsilon|)) \) a value \( P(d, \varepsilon) \) that is \( \varepsilon \)-close to \( P(d) \). Then the sets

\[
\{ c \in [0, 1] : \text{the language } \{d \in D | P(d) > c\} \text{ is in PSPACE} \}
\]

\[
\{ c \in [0, 1] : \text{the language } \{d \in D | P(d) \geq c\} \text{ is in PSPACE} \}
\]

are dense subsets of \([0, 1]\).

**Proof.** See Appendix H. \(\square\)

As a corollary we get:
Theorem 7. Let $\preceq \in \{\geq, >\}$. The sets

$$\{ c \in [0,1] : \text{the language } \{ \phi \in \Phi | c \phi \preceq c \} \text{ is in PSPACE} \}$$

$$\{ c \in [0,1] : \text{the language } \{ \phi \in \Phi | c' \phi \preceq c \} \text{ is in PSPACE} \}$$

$$\{ c \in [0,1] : \text{the language } \{ \phi \in \Phi | c'' \phi \preceq c \} \text{ is in PSPACE} \}$$

are dense subsets of $[0,1]$.

4. Conclusion

We have answered completely the question of the complexity of the problem if $\exists$ has strategy to achieve payoff 1 for all combinations of types of players. (For both players classical this is the classical QSAT problem.)

We have shown PSPACE-hardness of the question whether classical $\exists$ can make payoff greater than fixed $c$ when $\forall$ uses a probabilistic strategy. In the case of probabilistic $\exists$ and classical $\forall$ we need $c$ to be part of the input to get PSPACE-hardness. We have PSPACE-hardness result in the case of fixed $c$ when we ask whether $\exists$ can make payoff greater or equal to $c$. We have given $\Sigma^P_2$ lower bound for the question "$P(\phi) > c$?" in the case of both players being probabilistic and $c$ belonging to an input. We also indicate that for every mentioned problem it is possible to find a dense subset of thresholds for which the problem is in PSPACE.

Still many problems remain open. It would be nice to have a PSPACE-completeness result of the question "$P(\phi) > c$?" or "$P(\phi) \geq c$?" for some fixed $c$ ($c = \frac{1}{2}$ for instance) and for all combinations of types of players.

Also, the complexity of the problem of computing an approximation of game values (or exact values if possible) remains to be studied. This is the subject of an ongoing research.

Acknowledgement. The author wishes to express his thanks to prof. Damian Niwiński for many stimulating conversations.

References

We begin by proving implication \((9) \Rightarrow (10)\).

By \((9)\) there are \(x_1, x_3, \ldots, x_\kappa \in \{0, 1\}\) such that \(\phi(x_1, x_2, x_3, \ldots, x_\kappa) = 1\) for all \(x_2, x_4, \ldots, x_\kappa \in \{0, 1\}\). This clearly implies that \(P(\phi(x_1, X_2, x_3, \ldots, x_\kappa)) = 1\) for all random variables \(X_2, X_3, \ldots, X_\kappa\) with range \(\{0, 1\}\).

Implications \((10) \Rightarrow (11) \Rightarrow (13), (10) \Rightarrow (12) \Rightarrow (13)\) are obvious. We will show that \((13) \Rightarrow (9)\).

We claim that

\[
\forall X_2 \forall X_4 \ldots \forall X_\kappa \exists X_1 \exists X_3 \ldots \exists X_\kappa P(\phi) > \frac{1}{2^{\lceil n/2 \rceil}} \Rightarrow \exists x_1 \exists x_3 \ldots \exists x_\kappa \forall x_2 \forall x_4 \ldots \forall x_\kappa \phi.
\]

Suppose that \(\forall x_1 \forall x_3 \ldots \forall x_\kappa \exists x_2 \exists x_4 \ldots \exists x_\kappa \sim \phi\). It suffices to prove that

\[
\exists X_2 \exists X_4 \ldots \exists X_\kappa \forall X_1 \forall X_3 \ldots \forall X_\kappa P(\sim \phi) \geq \frac{1}{2^{\lceil n/2 \rceil}}.
\]

Let \(P(X_k = 1) = \frac{1}{2}, k = 2, 4, \ldots, \kappa\). For every \(x_1, x_3, \ldots, x_\kappa \in \{0, 1\}\) there exist \(x_k = x_k(x_1, x_3, \ldots, x_\kappa), k = 2, 4, \ldots, \kappa\), such that \(\phi(x_1, x_2, x_3, \ldots, x_\kappa) = 0\). Then for all random variables \(X_1, X_3, \ldots, X_\kappa\) we have

\[
P(\sim \phi(X_1, X_2, X_3, \ldots, X_\kappa)) = \sum_{(x_1, x_3, \ldots, x_\kappa) \in \{0, 1\}^{\kappa}} \prod_{j=1,3,\ldots,\kappa} P(X_j = x_j) \prod_{k=2,4,\ldots,\kappa} P(X_k = x_k)
\]

\[
\geq \sum_{(x_1, x_3, \ldots, x_\kappa) \in \{0, 1\}^{\lceil n/2 \rceil}} \prod_{j=1,3,\ldots,\kappa} P(X_j = x_j) \prod_{k=2,4,\ldots,\kappa} P(X_k = x_k(x_1, x_3, \ldots, x_\kappa))
\]

\[
= \left(\frac{1}{2}\right)^{\lceil n/2 \rceil} \sum_{(x_1, x_3, \ldots, x_\kappa) \in \{0, 1\}^{\lceil n/2 \rceil}} \prod_{j=1,3,\ldots,\kappa} P(X_j = x_j) = \frac{1}{2^{\lceil n/2 \rceil}}.
\]

We get \((13) \Rightarrow (9)\) from \((14)\) and following simple implication

\[
\exists X_1 \forall X_2 \exists X_4 \ldots \exists X_\kappa P(\phi) > 1 - \frac{1}{2^{\lceil n/2 \rceil}} \Rightarrow \forall X_2 \forall X_4 \ldots \forall X_\kappa \exists X_1 \exists X_3 \ldots \exists X_\kappa P(\phi) > 1 - \frac{1}{2^{\lceil n/2 \rceil}}.
\]

\(\square\)
The only nontrivial implication is (16). Let \( x_n \) be compact, and set 
\[ (16) \quad \exists x_1 \forall x_2 \exists x_3 \ldots \exists_n x_n \phi \]

\[ \Rightarrow \exists x_1 \forall x_2 \exists X_3 \ldots \exists_n x_n \phi \]

where \( x_n \) is compact if \( n \) is odd, and \( x_n \) if \( n \) is even.

**Proof.** The only nontrivial implication is (16) \( \Rightarrow \) (15). Assume formula (15) doesn’t hold. So there exists a winning strategy for \( \forall \) in (15). We prove that there is also a winning strategy for \( \forall \) in the game (16).

Let \( X_1 \) be chosen by \( \exists \) in the game (16). Let \( x_1 \in \{0, 1\} \) satisfy \( P(X_1 = x_1) \geq \frac{1}{2} \) and let \( x_1 \) be also the choice of \( \exists \) in the game (15). Let \( x_2 \in \{0, 1\} \) be chosen by \( \forall \), using a winning strategy, in the game (15). Let \( x_2 \) be also the choice of \( \forall \) also in the case of the game (16). And so on. Obviously \( \phi(x_1, x_2, x_3, \ldots, x_n) = 0 \). So in the case of the game (16)

\[
P(\sim \phi(X_1, x_2, X_3, \ldots, x_n)) \geq \prod_{j=1, 3, \ldots, \ell} P(X_j = x_j) \geq \frac{1}{2^{\lceil n/2 \rceil}} .
\]

\[ \square \]

**Appendix C.**

**Lemma 1.** Assume \( f : S \times T \to \mathbb{R} \) is a continuous map and \( S \times T \) are compact spaces. Then \( F \) defined by \( F(s) = \max_{t \in T} f(s, t) \) is also continuous.

**Proof.** By the compactness of \( T \) and the continuity of \( f \) for every \( s \) there exists \( t_s \) that maximizes \( f(s, t) \). Suppose the assertion of the lemma is false, and \( F \) is not continuous at some point \( s_0 \). There exists a positive \( \epsilon \) and a sequence \( s_n \) such that \( s_n \rightarrow s_0, \|s_n - s_0\| < \frac{1}{n} \) and \( |f(s_0, t_m) - f(s_n, t_m)| \geq \epsilon \). The product space \( S \times T \) is compact so \( (s_n, t_{s_n}) \rightarrow (s_0, t^*) \) for some subsequence \( (s_{n_k}, t_{s_{n_k}}) \) and some element \( t^* \). By the continuity of \( f \) we have \( f(s_{n_k}, t_{s_{n_k}}) \rightarrow f(s_0, t^*) \). By (i),(ii) and the definition of \( t_s \) the inequality (iii) \( f(s_0, t_m) - f(s_n, t^*) \geq \epsilon \) must hold. Let \( \delta > 0 \) be such, that \( \|s_0, (s_n, t_m) - (s_0, t^*)\| < \delta \) implies \( f(s_0, t_m) - f(s_0, t^*) < \epsilon/2 ; \) we can find such a \( \delta \) because \( f \) is continuous. So for every \( s_{n_k} \) such that \( |s_0 - s_{n_k}| < \delta \) we have \( f(s_0, t_{s_{n_k}}) - f(s_0, t_m) < \epsilon/2 \), contrary to (iii). \( \square \)

**Appendix D.**

**Theorem 3.** For every \( c \in \Gamma \setminus \{0\} \) the following problem is \( \text{PSPACE} \)-hard:

\[ \text{Given } \phi \text{ decide whether } \exists X_1 \forall x_2 \exists X_3 \ldots \exists_n x_n \phi \quad \text{P}(\phi) \geq c . \]

**Proof.** Let \( c = c' \) for some \( \varphi(w_1, \ldots, w_m) \in \Phi \). We can assume that \( m \) is even. Otherwise we can set \( \varphi'(w_1, \ldots, w_{m+1}) = \varphi(w_1, \ldots, w_m) \lor \varphi(w_{m+1}, \ldots) \). We shall reduce the problem \( \text{QSAT} \) to the presented problem.

For given Boolean formula \( \phi(v_1, \ldots, v_n) \), we set \( \phi'(w_1, \ldots, w_m, v_1, \ldots, v_n) = \varphi(w_1, \ldots, w_m) \land \varphi(v_1, \ldots, v_n) \), where \( v_i \neq w_j \) for all \( 1 \leq i \leq n, 1 \leq j \leq m \). We show that

\[ \exists x_1 \forall x_2 \ldots \exists_n x_n \phi \iff \exists Y_1 \forall y_2 \ldots \forall y_m \exists X_1 \forall x_2 \exists X_3 \ldots \exists_n x_n \phi \quad \text{P}(\phi') \geq c . \]

We prove the implication "\( \Rightarrow \)". \( \exists \) can use the winning strategy in the game \( \exists x_1 \forall x_2 \ldots \exists_n x_n \phi \) to win the game \( \exists Y_1 \forall y_2 \exists X_3 \ldots \exists_n x_n \phi \). So he can also win the
Proof. Let $c = c'_{\phi}$ for some $\varphi(w_1, \ldots, w_m) \in \Phi$. As in the proof of claim 3 we can assume that $m$ is even. We reduce the problem 2 to our problem.

For given Boolean formula $\phi(v_1, \ldots, v_n)$, where we can assume $v_i \neq v_j$ for all $1 \leq i \leq n, 1 \leq j \leq m$, we set $\phi' (w_1, \ldots, w_m, v_1, \ldots, v_n) = \varphi (w_1, \ldots, w_m) \lor \phi(v_1, \ldots, v_n)$. We prove the following equivalence

$$\exists x_1 \forall x_2 \ldots \exists \phi_n \exists \forall x_1 \forall X_2 \exists x_3 \ldots \exists \phi_n \varphi (\phi') > c.$$ 

For the proof of implication "$\implies$" notice that by assumption and theorem 2 player $\forall$ cannot win the game $\exists x_1 \forall X_2 \exists x_3 \ldots \exists \phi_n \varphi (\phi') > c$, for every $\forall i = 0$ and $\forall \phi$ as he can gain at most $P (\phi') = c_{\phi'} P (\phi) < c_{\phi'}$. We used the assumption that $c_{\phi'} \neq 0$ here. 

\appendix{E}

\textbf{Theorem 4.} For every $c \in \Gamma^\prime \setminus \{1\}$ the following problem is \textit{PSPACE}-hard:

Given $\phi$ decide whether $\exists x_1 \forall X_2 \exists x_3 \ldots \exists \phi_n \varphi (\phi) > c$.

\begin{proof}

For given Boolean formula $\phi(v_1, \ldots, v_n) \lor \varphi (w_1, \ldots, w_m) \lor \phi(v_1, \ldots, v_n)$, we can assume that $c = c'_{\phi}$ for some $\varphi (w_1, \ldots, w_m) \in \Phi$. As in the proof of claim 3 we can assume that $m$ is even. We reduce the problem 2 to our problem.

For given Boolean formula $\phi(v_1, \ldots, v_n)$, where we can assume $v_i \neq v_j$ for all $1 \leq i \leq n, 1 \leq j \leq m$, we set $\phi' (w_1, \ldots, w_m, v_1, \ldots, v_n) = \varphi (w_1, \ldots, w_m) \lor \phi(v_1, \ldots, v_n)$. We prove the following equivalence

$$\exists x_1 \forall x_2 \ldots \exists \phi_n \exists \forall x_1 \forall X_2 \exists x_3 \ldots \exists \phi_n \varphi (\phi') > c.$$ 

For the proof of implication "$\implies$" notice that by assumption and theorem 2 player $\forall$ cannot win the game $\exists x_1 \forall X_2 \exists x_3 \ldots \exists \phi_n \varphi (\phi') > c$. Thus $\exists$ can win it and so win the game $\exists x_1 \forall X_2 \ldots \forall \phi_n \exists x_1 \forall X_2 \exists x_3 \ldots \exists \phi_n \varphi (\phi') > c$ as he can attain

$$P (\phi') = P (\phi) + P (\phi) (1 - P (\phi)) > P (\phi) = c$$

because $P (\phi) > 0$ and $P (\phi) < 1$, as we assumed $c \neq 1$.

For the proof of implication "$\impliedby$" notice that if the formula $\exists x_1 \forall x_2 \ldots \exists \phi_n \varphi (\phi)$ is not true then $\forall$ can translate directly a winning strategy in the game $\exists x_1 \forall x_2 \ldots \exists \phi_n \varphi (\phi)$ to a winning strategy in the game $\exists x_1 \forall X_2 \exists x_3 \ldots \exists \phi_n \varphi (\phi) > 0$ and so win the game $\exists x_1 \forall X_2 \ldots \forall \phi_n \exists x_1 \forall X_2 \exists x_3 \ldots \exists \phi_n \varphi (\phi') > c$ as he can make $P (\phi') = P (\phi) + P (\phi) (1 - P (\phi)) = P (\phi) + 0 (1 - P (\phi)) = P (\phi) = c$. 

\appendix{F}

\textbf{Theorem 5.} $\forall \in \mathcal{A}$, $\forall \in \mathcal{B}$ and $\forall \in \mathcal{C}$.

\begin{proof}

Let $b = \sum_{i=1}^{n} b_i \frac{1}{2^i}$. Let $\phi = \bigvee_{k=1}^{n} \phi_k$ where

$$\phi_k = \begin{cases} 0 & \text{if } b_k = 0 \\ (v_{2k-1} \leftrightarrow v_{2k}) \land \bigwedge_{i=1}^{k-1} (v_{2i-1} \leftrightarrow v_{2i}) & \text{if } b_k = 1 \end{cases}$$

The proof of inclusions $\forall \subseteq \mathcal{A}$ and $\forall \subseteq \mathcal{B}$ will be completed by showing that $b = c_{\phi} = c_{\phi'}$.

\textbf{Claim F.1.} Let $\psi = v_1 \leftrightarrow v_2$ or $\phi = (v_1 \leftrightarrow v_2)$. Both the maps $p_1 \mapsto \min_{p_2 \in [0,1]} P_{p_1,p_2} (\phi)$ and $p_1 \mapsto \min_{p_2 \in [0,1]} P_{p_1,p_2} (\phi)$ attain their maximum values in $p_1 = \frac{1}{2}$. Moreover $P_{p_1,p_2} (\phi)$ doesn’t depend on $p_2$ when $p_1 = \frac{1}{2}$: $P_{p_1,p_2} (\phi) = P_{p_1,p_2} (\phi) = P_{p_3,p_2} (\phi) = \frac{1}{2}$ for every $p_2 \in [0,1]$. 

Proof. We proof the claim in the case $\phi = v_1 \leftrightarrow v_2$. Similar arguments apply to the case $\phi = \sim (v_1 \leftrightarrow v_2)$.

$$P_{p_1,p_2}(\phi) = p_1 p_2 + (1 - p_1)(1 - p_2) = (2p_1 - 1)p_2 + 1 - p_1.$$ 

$$\min_{p_2 \in [0,1]} P_{p_1,p_2}(\phi) = \min_{p_2 \in [0,1]} P_{p_1,p_2}(\phi) = \begin{cases} p_1 & \text{if } p_1 \leq \frac{1}{2} \\ 1 - p_1 & \text{if } p_1 > \frac{1}{2} \end{cases}$$

$$\max_{p_1 \in [0,1]} \min_{p_2 \in [0,1]} P_{p_1,p_2}(\phi) = \max_{p_1 \in [0,1]} \left\{ p_1 \begin{array}{ll} 1 - p_1 & \text{if } p_1 \leq \frac{1}{2} \\ 1 - p_1 & \text{if } p_1 > \frac{1}{2} \end{array} \right\} = \frac{1}{2}.$$ 

Observe $\min_{p_2 \in [0,1]} P_{p_1,p_2}(\phi) = \frac{1}{2}$ if and only $p_1 = \frac{1}{2}$ and $\min_{p_2 \in [0,1]} P_{p_1,p_2}(\phi) = \frac{1}{2}$ if and only $p_1 = \frac{1}{2}$. Moreover $P_{\frac{1}{2},p_2}(\phi) = P_{\frac{1}{2},p_2}(\sim \phi) = \frac{1}{2}$ for every $p_2 \in [0,1]$. $\square$

Assume $1 \leq k_1 < k_2 \leq n$. Since $\phi : (v_{2k_1-1} \leftrightarrow v_{2k_1})$ and $\phi : (v_{2k_1-1} \leftrightarrow v_{2k_1})$ the events $\{\phi : (X_1, \ldots, X_{2k_1}) = 1\}$ and $\{\phi : (X_1, \ldots, X_{2k_1}) = 1\}$ are disjoint. This gives

$$P_{p_1,\ldots,p_n}(\phi) = \sum_{k=1}^{n} P_{p_1,\ldots,p_{2k}}(\phi_k).$$

Since

$$P_{p_1,\ldots,p_{2k}}(\phi_k) = \begin{cases} 0 & \text{if } b_k = 0 \\ P_{p_{2k-1},p_{2k}}(v_{2k-1} \leftrightarrow v_{2k}) \prod_{i=1}^{k-1} P_{p_{2i-1},p_{2i}}(\sim (v_{2i-1} \leftrightarrow v_{2i})) & \text{if } b_k = 1 \end{cases}$$

we conclude from the claim F.1 that $\exists$ must pick $p_i = \frac{1}{2}$ for every $i = 1, 3, \ldots, 2k-1$ to make $P_{p_1,\ldots,x_n}(\phi)$ as maximal as possible. Thus

$$P_{p_1,\ldots,p_{2k}}(\phi_k) = \begin{cases} (\frac{1}{2})^k & \text{if } b_k = 1 \\ 0 & \text{if } b_k = 0 \end{cases}.$$ 

It follows easily that $c_0 = c_0' = \sum_{k=1}^{n} b_k \frac{1}{2^k}$.

As $\Gamma' = 1 - \Gamma''$ and $\Upsilon = \{1 - \gamma : \gamma \in \Upsilon\}$ we have $\Upsilon \subseteq \Gamma''$. By irrationality of $c_0 = F(\alpha^*)$ in example 3 (see also example 4) and irrationality of $c_0''$ in example 5, we get $\Upsilon \not\subseteq \Gamma$, $\Upsilon \not\subseteq \Gamma'$, and finally that $\Upsilon \not\subseteq \Gamma''$. This completes the proof of the theorem. $\square$

**APPENDIX G.**

**Theorem 6.** Let $\Delta_i = [0,1]$ or $\Delta_i = \{0,1\}$ for every $1 \leq i \leq n$. Given $\phi : (x_1, \ldots, x_n)$ and $\varepsilon > 0$, we can compute in $O(\log_2 |\phi| + n \log \log n + n \log_2 \varepsilon)$ space a number $\lambda'$ that is $\varepsilon$-close to $\lambda = \max_{p_1 \in \Delta_1, p_2 \in \Delta_2} \ldots \max p_1, p_2, \ldots, p_n(\phi)$. In particular, we can compute the approximation of game values $c_0$, $c_0'$, $c_0''$ within the bound just mentioned.

**Proof.** The value $\lambda'$ will be constructed as a game value of certain game. First we prove that $P_{p_1,\ldots,p_n}(\phi)$ is not too sensitive to small changes of inputs $p_1, \ldots, p_n$.

**Lemma G.1.** Suppose $|p_i - p_i'| \leq \frac{\varepsilon}{n}$ for every $1 \leq i \leq n$. Then

$$|P_{p_1,\ldots,p_n}(\phi) - P_{p_1',\ldots,p_n'}(\phi)| \leq \varepsilon.$$ 

**Proof.** It follows from definition of $P_{p_1,\ldots,p_n}(\phi)$ that it is a multilinear map with respect to all inputs $p_1, \ldots, p_n$. Let $D_i P_{p_1,\ldots,p_n}(\phi)$ be the $i$th partial derivative of $P_{p_1,\ldots,p_n}(\phi)$ at the point $p_i$. It does exist because $P_{p_1,\ldots,p_n}(\phi)$ is the linear map of $p_i$. Next we use the following theorem of Markov[3]:
Let $U : \mathbb{R} \to \mathbb{R}$ be a univariate polynomial of degree $d$ such that any real number $a_1 \leq p \leq a_2$ satisfies $u_1 \leq U(p) \leq u_2$. Then for all $a_1 \leq p \leq a_2$, the derivative of $U$ satisfies $|U'(p)| \leq \frac{2 \cdot |a_1 - a_2|}{a_2 - a_1}$.

Observe $0 \leq P_{p_1, \ldots, p_n}(\phi) \leq 1$ when $0 \leq p_1, \ldots, p_n \leq 1$. By Markov theorem we have $|D_i P_{p_1, \ldots, p_n}(\phi)| \leq M$ for all $0 \leq p_1, \ldots, p_n \leq 1$, where $M = 1$. So assuming $|p_i - p'_i| \leq \frac{\epsilon}{n}$ for all $i$ we get [1, page 14]

$$\left| P_{p_1, \ldots, p_n}(\phi) - P_{p'_1, \ldots, p'_n}(\phi) \right| \leq \frac{1}{n} \sqrt{nM} \sum_{i=1}^n (p_i - p'_i)^2 \leq \frac{1}{\sqrt{n}} \sqrt{n\epsilon^2} = \epsilon.$$ 

\[ \square \]

We shall prove that $\lambda$ can be approximated with bounded error.

**Lemma G.2.** Let $\Delta_i' \subseteq [0,1]$ ($1 \leq i \leq n$) be nonempty compact sets. Assume for all $p_i \in \Delta_i$ there exists $p'_i \in \Delta_i'$ such that $|p_i - p'_i| \leq \frac{\epsilon}{n}$ and for every $p_i \in \Delta_i'$ there exists $p_i \in \Delta_i$ such that $|p_i - p'_i| \leq \frac{\epsilon}{n}$. Then $|\lambda - \lambda'| \leq \epsilon$, where

$$\lambda' = \max \min \max \ldots \max_{p'_1 \in \Delta_1'} \ldots \max_{p'_n \in \Delta_n'} P_{p'_1, \ldots, p'_n}(\phi).$$

**Proof.** The main idea of the proof is to adopt the strategy used by $\exists$ in the game $A$ whose value is represented by $\lambda$ to the case of game $B$ whose value is represented by $\lambda'$ and similarly, to adopt the strategy used by $\exists$ in the case of the latter game to the case of the former game.

We show that $\lambda' \geq \lambda - \epsilon$. We present the strategy for $\exists$ in the game $B$ that enables him to gain $\hat{P}(\phi) \geq \lambda - \epsilon$.

Let $p'_1 \in \Delta'_1$ chosen by $\exists$ in game $B$ be $\frac{\epsilon}{n}$-close to $p_1$ chosen by him in $A$. Let $p_2$ chosen by $\forall$ in game $A$ be $\frac{\epsilon}{n}$-close to $p'_2$ chosen by him in $B$. And so on. Let $\lambda, \hat{\lambda}$ be the results of the games $A$ and $B$ correspondingly. Notice $\lambda \leq \hat{\lambda}$ because $\forall$ used in game $A$ probabilities adopted from game $B$ that need not to be the best choices of him in the $A$ case. Similarly $\hat{\lambda} \leq \lambda'$ because $\exists$ used in game $B$ probabilities adopted from game $A$ that need not to be the best choices of him in the $B$ case. By lemma G.1 we have $|\lambda - \hat{\lambda}| \leq \epsilon$, because $p_2$s and $p'_2$s used in games $A$ and $B$ respectively satisfy $|p_i - p'_i| \leq \frac{\epsilon}{n}$. So $\lambda' \leq \hat{\lambda} \leq \lambda - \epsilon \leq \lambda' \leq \lambda'$. In the same manner we can see that $\lambda \leq \lambda' - \epsilon$. 

\[ \square \]

Set $m = \lceil \log_2 n + \lceil \log_2 \epsilon \rceil \rceil$. Then $\frac{n}{2^m} \leq \epsilon$. Let $\Delta'_i = \Delta_i \cap \{ k \frac{1}{2^m} | k = 0, \ldots, 2^m \}$ for all $i$. We present how to compute $\lambda' = \max \min \max \ldots \max_{p'_1 \in \Delta'_1} \ldots \max_{p'_n \in \Delta'_n} P_{p'_1, \ldots, p'_n}(\phi)$ exactly. By lemma G.2 we have $|\lambda - \lambda'| \leq \epsilon$. Given $p_2$s we can compute $P_{p'_1, \ldots, p'_n}(\phi)$ precisely by the use of equation (2). Observe, that every $p'_i \in \Delta'_i$, $1 \leq i \leq n$, is of size $m$ bits i.e. only $m$ bits must be used to store the binary representation of $p'_i$. In order to $\lambda'$ be computed, we sum all the values $\prod_{i=1}^n p'_i(x_i)$ of size $nm$ bits for all tuples $(x_1, \ldots, x_n) \in \{0,1\}^n$ satisfying $\phi$. Notice that the Boolean value of $\phi(x_1, \ldots, x_n)$ can be determined in space $O(\log_2 |\phi|)$ and that the sum of two values of size $nm$ bits is also the value of size $nm$ bits. So we can compute $\lambda'$ in space $O(\log_2 |\phi| + nm)$ because the stack $\max \min \max \ldots \max_{p'_1 \in \Delta'_1} \ldots \max_{p'_n \in \Delta'_n}$ of currently analyzed values of $p'_1, p'_2, p'_3, \ldots, p'_n$ can also be stored in space of size $O(nm)$. As $O(\log_2 |\phi| + nm) = O(\log_2 |\phi| + n \log_2 n + n |\log_2 \epsilon|)$, the proof of theorem is complete. 

\[ \square \]
APPENDIX H.

**Lemma 2.** Let $D \subseteq \Sigma^*$ be a language over a finite alphabet $\Sigma$, $|\Sigma| \geq 2$, and let $P$ be a map $P : D \to [0,1]$. Assume for given $d \in D$ we can compute in space $O(\text{Poly}(|d|, |\log \varepsilon|))$ a value $P(d,\varepsilon)$ that is $\varepsilon$-close to $P(d)$. Then the sets

\[
\{c \in [0,1] : \text{the language } \{d \in D | P(d) > c\} \text{ is in PSPACE}\}
\]

\[
\{c \in [0,1] : \text{the language } \{d \in D | P(d) \geq c\} \text{ is in PSPACE}\}
\]

are dense subsets of $[0,1]$.

**Proof.** To make the proof easier, we assume that $|\Sigma| = 2$. This is no loss of generality, as we have a linear time computable map $t : \Sigma^* \to [0,1]^*$ ($|\Sigma| \geq 2$), with $t^{-1}$ also being linear time computable. Fix nonempty interval $[a,b] \subseteq [0,1]$. There is no loss of generality in assuming that numbers $a,b$ are finitely representable rational numbers. We shall describe Turing machine $A$ which given $n \in \mathbb{N}$ computes $a_n, b_n, \varepsilon_n$. Construction of $A$ ensures that $[a_0, b_0] = [a,b]$ and $[a_n, b_n] \subset [a_{n-1}, b_{n-1}]$ ($n > 0$) so by Cantor theorem there exists $c \in \bigcap_{i=0}^{\infty} [a_i, b_i]$. Especially if $b_n - a_n \to 0$ then such $c$ is unique. We will show that for this $c$ the problem $\{d \in D | P(d) > c\}$ is in PSPACE. Let us denote by $D_n$ the set of all inputs in $D$ of size $n$. Notice, that $|D_n| \leq 2^n$, $n \geq 0$. For every $d \in D_n$ we will also have

\[
[a_n, b_n] \cap [P(d, \varepsilon_n) - \varepsilon_n, P(d, \varepsilon_n) + \varepsilon_n] = \emptyset.
\]

As $P(d, \varepsilon_n)$ is $\varepsilon_n$-close to $P(d)$ we have $P(d) \in [P(d, \varepsilon_n) - \varepsilon_n, P(d, \varepsilon_n) + \varepsilon_n]$. So $P(d) > c$ iff $P(d, \varepsilon_n) > b_n$ and following Turing machine can be used to decide the problem whether $P(d) > c$ holds, and equivalently whether $P(d) \geq c$ holds.

---

**Turing machine $G$**

**Input:** $d \in D$

Let $n$ be the size of $d$. Set $(a_n, b_n, \varepsilon_n) := A(n)$. If $P(d, \varepsilon_n) > b_n$ then output true, otherwise output false.

---

**Claim H.1.** Let $\Delta_1, \ldots, \Delta_{2^n}$ ($n \geq 0$) be nonempty closed intervals contained in interval $\Delta$. Let $|\Delta|$ denote the length of interval $\Delta$ and let $\text{Int} \Delta$ denote the interior of $\Delta$. Assume $|\Delta_i| \leq \frac{1}{2^{n+1}} |\Delta|$ for all $i$. Then there exists an interval $\Delta' \subseteq \Delta$ satisfying $|\Delta'| \geq \frac{1}{2^{n+1}} |\Delta|$ and $\text{Int} \Delta' \cap \text{Int} \Delta_i = \emptyset$ for all $i$.

**Proof.** Without the loss of the generality we can assume that the interiors of $\Delta_i$s are pairwise disjoint. We can also assume that intervals are sorted; $x \in \Delta_i, y \in \Delta_{i+1}$ implies $x \leq y$. Let $\Delta_0, \ldots, \Delta_{2^n}$ be intervals such that $\Delta_0 \cup \Delta_1 \cup \Delta'_1 \cup \ldots \cup \Delta_{2^n-1} \cup \Delta_{2^n} \cup \Delta'_{2^n} = \Delta$, $\text{Int} \Delta_i \cap \text{Int} \Delta_j = \emptyset$ for $i \neq j$ and $\text{Int} \Delta_i \cap \text{Int} \Delta_j = \emptyset$ for all $i, j$. Then $|\Delta_0| + |\Delta'_1| + \ldots + |\Delta_{2^n}| \geq |\Delta| - 2^n \frac{1}{2^{n+1}} |\Delta| = \frac{1}{2} |\Delta|$. So there exists $\Delta'$ with $|\Delta'| \geq \frac{1}{2} |\Delta| / (2^n + 1) \geq \frac{1}{2^{n+1}} |\Delta|$.

---

Let $\Delta = [a_{n-1}, b_{n-1}]$. For $d \in D_n$ the length of interval $\Delta_d = [P(d, \varepsilon_n) - \varepsilon_n, P(d, \varepsilon_n) + \varepsilon_n]$ will be $|\Delta_d| = 2\varepsilon_n = 2\frac{1}{2^{n+1}} |\Delta| = \frac{1}{2^{n+1}} |\Delta|$. By the use of the Claim H.1 we can state that there is $\Delta' \subset \Delta$ such that $\text{Int} \Delta' \cap \text{Int} \Delta_d = \emptyset$ for all $d \in D_n$ and $|\Delta'| \geq \frac{1}{2^{n+1}} |\Delta|$. Let $\Delta' = [s,t]$. To avoid the possibility that $P(d_1) = a_n$ or $P(d_2) = b_n$ we shall set $(a_n, b_n) := (s + \frac{1}{4} (t-s), t - \frac{1}{4} (t-s))$ instead of $(a_n, b_n) := (s,t)$.

---

**Turing machine $A$**

**Input:** $n$
If \( n = 0 \) then we set \((a_{-1}, b_{-1}) := (a, b)\) and \(\varepsilon_0 := \frac{1}{2^2} (b_{-1} - a_{-1})\), otherwise we set \((a_{n-1}, b_{n-1}, \varepsilon_{n-1}) := A(n-1)\) and \(\varepsilon_n := \frac{1}{2^{n+2}} (b_{n-1} - a_{n-1})\). To make description of \(A\) easier we introduce artificial inputs \(\alpha_n, \beta_n\) of size \(n\) and we assume \(P(\alpha_n, \varepsilon_n) = a_{n-1}\) and \(P(\beta_n, \varepsilon_n) = b_{n-1}\): \(D'_n := D_n \cup \{\alpha_n, \beta_n\}\). We look for the only pair \((d_1, d_2) \in D'_n \times D'_n\) satisfying

1. \(P(d_1, \varepsilon_n) < P(d_2, \varepsilon_n)\),
2. there is no \(d \in D_n\) such that \(P(d, \varepsilon_n) \in (P(d_1, \varepsilon_n), P(d_2, \varepsilon_n))\),
3. \(P(d_2, \varepsilon_n) - P(d_1, \varepsilon_n) \geq P(d'_2, \varepsilon_n) - P(d'_1, \varepsilon_n)\) for all \((d'_1, d'_2) \in D_n \times D_n\) satisfying (1) and (2),
4. \(P(d_1, \varepsilon_n) \leq P(d'_1, \varepsilon_n)\) for all \((d'_1, d'_2) \in D_n \times D_n\) satisfying (1), (2) and (3).

Condition (4) is intended to assure the uniqueness of a pair \((d_1, d_2)\), we are looking for. If \(d_1 \in D_n\) then we set \(s := P(d_1, \varepsilon_n) + \varepsilon_n\), otherwise we set \(s := a_{n-1}\). Similarly, if \(d_2 \in D_n\) then we set \(t := P(d_2, \varepsilon_n) - \varepsilon_n\), otherwise we set \(t := b_{n-1}\). After that we set \((a_n, b_n, \varepsilon_n) := (s + \frac{1}{2} (t - s), t - \frac{1}{2} (t - s))\).

\begin{align*}
\text{Output: } a_n, b_n, \varepsilon_n
\end{align*}

We shall have established that \(A\) runs in \(O(\text{Poly}(n))\)-space and so does \(G\) if we prove that \(|\log \varepsilon_n| = O(n^2)\), because then given \(\varepsilon_n\) and \(d \in D_n\) we can compute \(P(d, \varepsilon_n)\) in space \(O(\text{Poly}(n, O(n^2))) = O(\text{Poly}(n))\). Since \(b_n - a_n \geq \frac{1}{2^{n+3}} (b_{n-1} - a_{n-1})\), it follows that \(b_n - a_n \geq \left( \prod_{i=0}^{n-1} \frac{1}{2^{i+3}} \right) (b - a)\), when \(n \geq 0\). Hence

\begin{align*}
\log_2 \varepsilon_n &= \log_2 \frac{1}{2^{n+3}} (b_{n-1} - a_{n-1}) \geq \log_2 \frac{1}{2^{n+3}} \left( \prod_{i=0}^{n-1} \frac{1}{2^{i+3}} \right) (b - a) \\
&= \log_2 (b - a) - \left( (n + 2) + \sum_{i=0}^{n-1} (i + 3) \right).
\end{align*}

We thus get \(- \log_2 \varepsilon_n \leq - \log_2 (b - a) + \left( \frac{1}{2} n^2 + \frac{7}{2} n + 2 \right) = O(n^2)\). \(\square\)